

# Publications mathématiques de Besançon

## ALGÈBRE ET THÉORIE DES NOMBRES

Laurent Berger

**Lubin's conjecture for full  $p$ -adic dynamical systems**

2016, p. 19-24.

<[http://pmb.cedram.org/item?id=PMB\\_2016\\_\\_\\_\\_19\\_0](http://pmb.cedram.org/item?id=PMB_2016____19_0)>

© Presses universitaires de Franche-Comté, 2016, tous droits réservés.

L'accès aux articles de la revue « Publications mathématiques de Besançon » (<http://pmb.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://pmb.cedram.org/legal/>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

*Publication éditée par le laboratoire de mathématiques  
de Besançon, UMR 6623 CNRS/UFC*

cedram

*Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

LUBIN'S CONJECTURE FOR FULL  $p$ -ADIC DYNAMICAL SYSTEMS*by*

Laurent Berger

**Abstract.** — We give a short proof of a conjecture of Lubin concerning certain families of  $p$ -adic power series that commute under composition. We prove that if the family is *full* (large enough), there exists a Lubin-Tate formal group such that all the power series in the family are endomorphisms of this group. The proof uses ramification theory and some  $p$ -adic Hodge theory.

**Résumé.** — (*La conjecture de Lubin pour les systèmes dynamiques  $p$ -adiques pleins*) Nous donnons une démonstration courte d'une conjecture de Lubin concernant certaines familles de séries formelles  $p$ -adiques qui commutent pour la composition. Nous montrons que si la famille est *pleine* (assez grosse), il existe un groupe formel de Lubin-Tate tel que toutes les séries de la famille sont des endomorphismes de ce groupe. La démonstration utilise la théorie de la ramification et un peu de théorie de Hodge  $p$ -adique.

## Introduction

Let  $K$  be a finite extension of  $\mathbf{Q}_p$ , and let  $O_K$  be its ring of integers. In [5], Lubin studied  $p$ -adic dynamical systems, namely families of elements of  $T \cdot O_K[[T]]$  that commute under composition, and remarked that “experimental evidence seems to suggest that for an invertible series to commute with a noninvertible series, there must be a formal group somehow in the background”. This observation has motivated the work of a number of people (Hsia, Laubie, Li, Movaheddi, Salinier, Sarkis, Specter, ...) who proved various results in that direction. The purpose of this note is to give a proof of a special case of the above observation, which is referred to as “Lubin’s conjecture” in §3.1 of [7]. Let us consider a family  $F$  of commuting power series  $F(T) \in T \cdot O_K[[T]]$ . We say that such a family is *full* if for all  $\alpha \in O_K$  there exists  $F_\alpha(T) \in F$  such that  $F_\alpha(0) = \alpha$  and if  $\text{wided}(F_\pi(T)) = q$ , where  $\text{wided}(F(T))$  denotes the Weierstrass degree of  $F(T)$ ,  $\pi$  is any uniformizer of  $O_K$  and  $q$  is the cardinality of the residue field of  $O_K$ .

**Mathematical subject classification (2010).** — 11S82, 11S15, 11S20, 11S31, 13F25, 13F35, 14F30.

**Key words and phrases.** —  $p$ -adic dynamical system, Lubin-Tate formal group,  $p$ -adic Hodge theory.

**Theorem.** — *If  $F$  is a full family of commuting power series, there exists a Lubin-Tate formal group  $G$  such that  $F_\alpha(T) \in \text{End}(G)$  for all  $\alpha \in \mathcal{O}_K$ .*

This result already appears as Theorem 2 of [4]. Our proof is however considerably shorter than that of *ibid.*, and does not use the theory of the field of norms. It is very similar to that of the main result of [8], which treats the case  $K = \mathbf{Q}_p$ . The main ingredients are ramification theory and some  $p$ -adic Hodge theory. In order to simplify the use of  $p$ -adic Hodge theory, we assume that  $K$  is a Galois extension of  $\mathbf{Q}_p$ .

### 1. $p$ -adic dynamical systems

In this note, we consider a set  $F = \{F_\alpha(T)\}_{\alpha \in \mathcal{O}_K}$  of power series  $F_\alpha(T) \in T \cdot \mathcal{O}_K[[T]]$  such that  $F_\alpha(0) = \alpha$  and  $F_\alpha \circ F_\beta(T) = F_\beta \circ F_\alpha(T)$  whenever  $\alpha, \beta \in \mathcal{O}_K$ . Recall that  $\pi$  is a uniformizer of  $\mathcal{O}_K$ , and that  $q$  is the cardinality of the residue field  $k$  of  $\mathcal{O}_K$ . If  $F(T)$  is a power series and  $n > 0$ , we denote by  $F^{(n)}(T)$  the  $n$ -th fold iteration  $F \circ \dots \circ F(T)$ . If  $F(T)$  has an inverse for the composition, this definition extends to  $n \in \mathbf{Z}$ . Recall that the *Weierstrass degree*  $\text{wdeg}(F(T))$  of  $F(T) = \sum_{i=1}^+ f_i T^i \in T \cdot \mathcal{O}_K[[T]]$  is the smallest integer  $i$  such that  $f_i \in \mathcal{O}_K^\times$ . By the Weierstrass preparation theorem, if  $\text{wdeg}(F) = +$ , then  $F$  has  $\text{wdeg}(F)$  zeroes in  $\mathfrak{m}_{\mathbf{C}_p}$ .

**Proposition 1.1.** — *There exists a power series  $G(T) \in T \cdot k[[T]]$  and an integer  $d > 1$  such that  $G(0) \in k^\times$  and  $F_\pi(T) = G(T^{p^d})$ .*

*Proof.* — See (the proof of) theorem 6.3 and corollary 6.2.1 of [5].

**Proposition 1.2.** — *There exists a power series  $L_F(T) \in K[[T]]$  such that*

1.  $L_F(T) = T + \mathcal{O}(T^2)$ ;
2.  $L_F(T)$  converges on the open unit disk;
3.  $L_F \circ F_\alpha(T) = \alpha \cdot L_F(T)$  for all  $\alpha \in \mathcal{O}_K$ .

*Proof.* — See propositions 1.2 and 2.2 of [5] for the construction of a unique power series  $L_F(T)$  that satisfies (1), (2) and (3) for  $\alpha$  a uniformizer of  $\mathcal{O}_K$ . If  $\beta \in \mathcal{O}_K \setminus \{0\}$ , then  $\beta^{-1} \cdot L_F \circ F_\beta$  also satisfies (1), (2) and (3) for  $\alpha$  as above, so that  $L_F \circ F_\beta(T) = \beta \cdot L_F(T)$  for all  $\beta \in \mathcal{O}_K$ .

The hypothesis that  $F$  is full implies that  $p^d = q$ , so that  $\text{wdeg}(F_\pi(T)) = q$ . For  $n > 1$ , let  $\mathfrak{m}_n$  denote the set of  $u \in \mathfrak{m}_{\mathbf{C}_p}$  such that  $F_\pi^{(n)}(u) = 0$  and  $F_\pi^{(n-1)}(u) \neq 0$  and let  $\mathfrak{m}_n = \bigcup_{n>1} \mathfrak{m}_n$ . Proposition 1.1 implies that  $F_\pi(T)/\pi$  is a unit of  $\mathcal{O}_K[[T]]$ , so that the roots of  $F_\pi^{(n)}(T)$  are simple for all  $n > 1$ . The set  $\mathfrak{m}_n$  therefore has  $q^{n-1}(q-1)$  elements.

The series  $F_\alpha(T)$  is invertible if  $\alpha \in \mathcal{O}_K^\times$  so that in this case,  $F_\alpha(z) = 0$  if and only if  $z = 0$ . If  $u \in \mathfrak{m}_n$  and  $\alpha \in \mathcal{O}_K^\times$ , then  $F_\pi^{(n)} \circ F_\alpha(u) = F_\alpha \circ F_\pi^{(n)}(u) = 0$  and  $F_\pi^{(n-1)} \circ F_\alpha(u) = F_\alpha \circ F_\pi^{(n-1)}(u) = 0$  so that the action of  $F_\alpha(T)$  permutes the elements of  $\mathfrak{m}_n$ .

Let  $K_n = K(\mathfrak{m}_n)$ , so that  $\mathfrak{m}_i \subset K_n$  if  $i \leq n$ , and let  $K = \bigcup_{n>1} K_n$ . If  $\alpha \in \mathcal{O}_K^\times$ , let  $n(\alpha)$  be the largest integer  $n > 0$  such that  $\alpha \equiv 1 + \pi^n \pmod{\mathcal{O}_K}$ .

**Proposition 1.3.** — *If  $n > 1$  and  $u \in \mathfrak{m}_n$ , then*

1.  $F_\alpha(u) = u$  if and only if  $n(\alpha) > n$ ;
2. If  $n(\alpha) = n$ , then  $\text{wided}(F_\alpha(T) - T) = q^n$ ;
3.  $n = \{F_\alpha(u)\}_\alpha O_K^\times$ .

*Proof.* — If  $n = 1$  and  $F_\alpha(u) = u$ , then  $u$  is a root of  $F_\alpha(T) - T = (\alpha - 1)T + O(T^2)$ , so that  $\alpha - 1 \in \pi O_K$ . This implies that  $\{F_\alpha(u)\}_\alpha O_K^\times$  has at least  $q - 1$  distinct elements. These elements are all roots of  $F_\pi(T)/T$ , whose  $\text{wided}$  is  $q - 1$ , so  $\{F_\alpha(u)\}_\alpha O_K^\times$  has precisely  $q - 1$  elements. These elements all have valuation  $1/(q - 1)$ , and if  $n(\alpha) = 1$ , the Newton polygon of  $F_\alpha(T) - T$  starts at the point  $(1, 1)$ , so that it can have only one segment, and  $\text{wided}(F_\alpha(T) - T) = q$ . This implies the proposition for  $n = 1$ .

Assume now that the proposition holds up to some  $n > 1$  and take  $u \in O_K^\times$ . If  $n(\alpha) \leq n$ , then  $F_\alpha(T) - T$  has at most  $q^n$  roots by (2), contained in  $O_K^\times \dots O_K^\times$  by (1). Therefore  $F_\alpha(u) = u$  implies  $n(\alpha) > n + 1$ . The set  $\{F_\alpha(u)\}_\alpha O_K^\times$  therefore has at least  $q^n(q - 1)$  distinct elements, all of them roots of  $F_{\pi^{n+1}}(T)/F_{\pi^n}(T)$ .

This implies that  $\{F_\alpha(u)\}_\alpha O_K^\times$  has exactly  $q^n(q - 1)$  elements. If  $n(\alpha) = n + 1$ , the Newton polygon of  $F_\alpha(T) - T$  starts at the point  $(1, n + 1)$ , with  $n + 1$  segments of height one and slopes  $-1/q^k(q - 1)$  with  $0 \leq k \leq n$ , so that it reaches the point  $(q^{n+1}, 0)$  and hence  $\text{wided}(F_\alpha(T) - T) = q^{n+1}$ . This implies the proposition for  $n + 1$ .

**Corollary 1.4.** — *The field  $K$  is an abelian totally ramified extension of  $K$ , and if  $g \in \text{Gal}(K/K)$ , there is a unique  $\eta(g) \in O_K^\times$  such that  $g(u) = F_{\eta(g)}(u)$  for all  $u \in O_K^\times$ . The map  $\eta : \text{Gal}(K/K) \rightarrow O_K^\times$  is an isomorphism.*

*Proof.* — Take  $u \in O_K^\times$  and  $\alpha \in O_K^\times$ . As we have seen above,  $F_\alpha(u) \in O_K^\times$ , so that the map  $u \mapsto F_\alpha(u)$  induces a field automorphism of  $K(u)$ . By (3) of Proposition 1.3, this implies that  $K_n = K(u)$  and that every element of  $\text{Gal}(K_n/K)$  comes from  $u \mapsto F_\alpha(u)$  for some  $\alpha \in O_K^\times$ . The extension  $K_n/K$  is therefore abelian, and so is  $K/K$ . Since  $K_n = K(u)$ , the extension  $K_n/K$  is totally ramified, and so is  $K/K$ .

The map  $\eta$  is surjective because every  $F_\alpha(T)$  gives rise to an automorphism of  $K$ , and it is injective because if  $\eta(g) = 1$ , then  $g(u) = u$  for all  $u \in O_K^\times$  so that  $g = 1$ .

In order to prove our main theorem, we study the  $p$ -adic periods of  $\eta$ . Corollary 1.4 and local class field theory imply that the extension  $K/K$  is attached to a uniformizer  $\varpi$  of  $O_K$ . Let  $\chi_\varpi : G_K \rightarrow O_K^\times$  denote the corresponding Lubin-Tate character.

## 2. $p$ -adic Hodge theory

Let  $R$  be the  $p$ -adic completion of  $\varinjlim_{n>1} O_K[[X_n]]$  where  $O_K[[X_n]]$  is seen as a subring of  $O_K[[X_{n+1}]]$  via the identification  $X_n = F_\pi(X_{n+1})$  (this ring is defined in [8], where it is denoted by  $A$ ). We define an action of  $G_K$  on  $R$  by  $g(H(X_n)) = H(F_{\eta(g)}(X_n))$ . This is well-defined since  $F_\pi \circ F_{\eta(g)}(T) = F_{\eta(g)} \circ F_\pi(T)$ . We have  $R/\pi R = \varinjlim_{n>1} k[[X_n]]$ .

**Lemma 2.1.** — *The ring  $R/\pi R$  is perfect.*

*Proof.* — Let  $G(T)$  be as in Lemma 1.1. The fact that  $X_n = F_\pi(X_{n+1})$  implies that  $G^n(X_n) = G^{n+1}(X_{n+1})^q$ . Since  $G(0) = k^\times$ , we have  $k[[T]] = k[[G(T)]]$  and therefore

$$R/\pi R = \varinjlim_{G^n(X_n) = G^{n+1}(X_{n+1})^q} k[[G^n(X_n)]]$$

is perfect.

Let  $\tilde{\mathbf{E}}^+ = \varinjlim_{-x \ x^q} \mathcal{O}_{\mathbf{C}_p}/\pi$ . Choose a sequence  $\{u_n\}_{n>1}$  with  $u_n \in \mathbf{C}_p$  and  $F_\pi(u_{n+1}) = u_n$ . This sequence gives rise to a map  $i : R/\pi R \rightarrow \tilde{\mathbf{E}}^+$ , determined by the requirement that  $i(X_n) = (G^{-1}(\bar{u}_n), G^{-2}(\bar{u}_{n+1}), \dots)$ . The definition of the action of  $G_K$  on  $R$  and Corollary 1.4 imply that  $i$  is  $G_K$ -equivariant.

**Lemma 2.2.** — *The map  $i : R/\pi R \rightarrow \tilde{\mathbf{E}}^+$  is injective.*

*Proof.* — It is enough to show that  $i : k[[X_n]] \rightarrow \tilde{\mathbf{E}}^+$  is injective. If it was not, there would be a nonzero polynomial  $P(T) \in k[[T]]$  such that  $P(i(X_n)) = 0$ , and then  $i(X_n) = (G^{-1}(\bar{u}_n), G^{-2}(\bar{u}_{n+1}), \dots)$  would belong to  $\bar{\mathbf{F}}_p$ , which is clearly not the case.

Let  $K_0 = \mathbf{Q}_p^{\text{unr}} \subset K$  and let  $\tilde{\mathbf{A}}^+ = \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} W(\tilde{\mathbf{E}}^+)$  (see [2]; note that  $\tilde{\mathbf{A}}^+$  usually denotes  $W(\tilde{\mathbf{E}}^+)$ , and is denoted by  $A_{\text{inf}}$  in *ibid.*). We have  $R = \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} W(R/\pi R)$  since  $R$  is a strict  $\pi$ -ring, and by the functoriality of Witt vectors, the map  $i$  extends to an injective and  $G_K$ -equivariant map  $i : R \rightarrow \tilde{\mathbf{A}}^+$ . We write  $x$  instead of  $i(X_1) \in \tilde{\mathbf{A}}^+$ . The  $G_K$ -equivariance of  $i$  implies that  $g(x) = F_{\eta(g)}(x)$ .

Let  $\mathbf{B}_{\text{cris}}^+$  and  $\mathbf{B}_{\text{dR}}$  be some of Fontaine's rings of periods. Recall that  $\mathbf{B}_{\text{dR}}$  is a field, that there is a Frobenius map  $\varphi$  on  $\mathbf{B}_{\text{cris}}^+$ , a filtration  $\{\text{Fil}^i \mathbf{B}_{\text{dR}}\}_i \subset \mathbf{Z}$  on  $\mathbf{B}_{\text{dR}}$ , and an injective map  $K \otimes_{K_0} \mathbf{B}_{\text{cris}}^+ \rightarrow \mathbf{B}_{\text{dR}}$ . There is also an action of  $G_K$  on  $\mathbf{B}_{\text{cris}}^+$  and  $\mathbf{B}_{\text{dR}}$  compatible with the above structure, and  $\mathbf{B}_{\text{dR}}^{G_K} = K$ . Let  $\varphi_q = \varphi^f$  on  $\mathbf{B}_{\text{cris}}^+$ , where  $q = p^f$ , extended by  $K$ -linearity to  $K \otimes_{K_0} \mathbf{B}_{\text{cris}}^+$ . We refer to [2] and [3] for the properties of these objects. Let  $\tau \in \text{Gal}(K/\mathbf{Q}_p)$ . If  $\tau \in \mathbf{Z}$ , choose a  $n(\tau) \in \mathbf{Z}_{>0}$  such that  $\tau|_{K_0} = \varphi^{n(\tau)}$ . The map  $\tau \cdot \varphi^{n(\tau)} : K \otimes_{K_0} \mathbf{B}_{\text{cris}}^+ \rightarrow K \otimes_{K_0} \mathbf{B}_{\text{cris}}^+$  is then well-defined and commutes with  $\varphi_q$  and the action of  $G_K$ .

We say that a character  $\lambda : G_K \rightarrow \mathcal{O}_K^\times$  is *crystalline positive* if there exists a nonzero  $z \in K \otimes_{K_0} \mathbf{B}_{\text{cris}}^+$  such that  $g(z) = \lambda(g) \cdot z$  for all  $g \in G_K$ . The following proposition summarizes the input that we need from the  $p$ -adic Hodge theory of characters.

**Proposition 2.3.** — *A character  $\lambda : G_K \rightarrow \mathcal{O}_K^\times$  that factors through  $\text{Gal}(K/K)$  is crystalline positive if and only if  $\lambda = \prod_{\tau} \tau \cdot \chi_{\varpi}^{h_{\tau}}$  with  $h_{\tau} \in \mathbf{Z}_{>0}$ .*

*If  $t_{\varpi} \in K \otimes_{K_0} \mathbf{B}_{\text{cris}}^+$  is such that  $g(t_{\varpi}) = \chi_{\varpi}(g) \cdot t_{\varpi}$  for all  $g \in G_K$ , then  $t_{\varpi} \in \text{Fil}^1 \mathbf{B}_{\text{dR}}$  and  $\varphi_q(t_{\varpi}) = \varpi \cdot t_{\varpi}$ .*

*Sketch of proof.* — If  $\lambda : G_K \rightarrow \mathcal{O}_K^\times$  is a crystalline positive character and  $h_{\tau} \in \mathbf{Z}_{>0}$  denotes the Hodge-Tate weight of  $\lambda$  at  $\tau \in \mathbf{Z}$ , then  $\lambda \cdot \prod_{\tau} \tau \cdot \chi_{\varpi}^{-h_{\tau}}$  is crystalline and has Hodge-Tate weight zero at all  $\tau \in \mathbf{Z}$  so that it is unramified, and therefore trivial if  $\lambda$  factors through  $\text{Gal}(K/K)$ , since  $K/K$  is totally ramified.

Let  $\omega_E$  and  $t_E$  be the elements constructed in §9.2 and §9.3 of [1] (with  $E = K$  and  $\pi_E = \varpi$ ). We have  $t_E \in K \otimes_{K_0} \mathbf{B}_{\text{cris}}^+$  and  $\varphi_q(t_E) = \varpi \cdot t_E$  and  $t_E \in \text{Fil}^1 \mathbf{B}_{\text{dR}}$  (proposition 9.10 of *ibid.*). If  $g \in G_K$ , then (in the notation of *ibid.* and where  $[\cdot]_{\text{LT}}$  denotes the endomorphisms

of the Lubin-Tate group attached to  $\varpi$ ) we have  $g(\omega_E) = [\chi_\varpi(g)]_{\text{LT}}(\omega_E)$  and therefore  $g(t_E) = g(F_E(\omega_E)) = F_E(g(\omega_E)) = F_E([\chi_\varpi(g)]_{\text{LT}}(\omega_E)) = \chi_\varpi(g) \cdot F_E(\omega_E) = \chi_\varpi(g) \cdot t_E$  since  $F_E$  is the logarithm of the Lubin-Tate group attached to  $\varpi$ . If  $t_\varpi \in K_{K_0} \mathbf{B}_{\text{cris}}^+$  is such that  $g(t_\varpi) = \chi_\varpi(g) \cdot t_\varpi$  for all  $g \in G_K$ , then  $t_\varpi/t_E \in \mathbf{B}_{\text{dR}}^{G_K} = K$ , and this implies the rest of the proposition.

Recall that  $L_F(T) \in K[[T]]$  is the logarithm attached to  $F$ . Since  $L_F(T)$  converges on the open unit disk, we can view  $L_F(x)$  as an element of  $K_{K_0} \mathbf{B}_{\text{cris}}^+$ .

**Proposition 2.4.** — *The character  $\eta : G_K \rightarrow O_K^\times$  is crystalline positive.*

*Proof.* — If  $g \in G_K$ , then  $g(L_F(x)) = L_F(g(x)) = L_F(F_{\eta(g)}(x)) = \eta(g) \cdot L_F(x)$ .

**Corollary 2.5.** — *We have  $L_F(x) = \beta \cdot \prod_{\tau} (\tau \cdot \varphi^{n(\tau)})(t_\varpi)^{h_\tau}$  where  $h_\tau \in \mathbf{Z}_{>0}$  and  $\beta \in K^\times$ .*

*Proof.* — This follows from the facts that  $\eta = \prod_{\tau} \tau \cdot \chi_\varpi^{h_\tau}$ , that  $\chi_\varpi(g) = g(t_\varpi)/t_\varpi$  and that  $\mathbf{B}_{\text{dR}}^{G_K} = K$ .

**Proposition 2.6.** — *We have  $\varphi_q(L_F(x)) = \mu \cdot L_F(x)$  for some  $\mu \in \pi O_K$ .*

*Proof.* — Corollary 2.5 and Proposition 2.3 imply the proposition with  $\mu = \prod_{\tau} \tau(\varpi)^{h_\tau}$ , and not all  $h_\tau$  can be equal to 0 since  $\eta = 1$ .

**Corollary 2.7.** — *We have  $\varphi_q(x) = F_\mu(x)$ .*

*Proof.* — Proposition 2.6 implies that  $L_F(\varphi_q(x)) = L_F(F_\mu(x))$ . We would like to apply  $L_F^{-1}(T)$  but we have to mind the convergence and need to proceed as follows. Since  $\eta$  is nontrivial, there is a  $\tau$  such that  $h_{\tau^{-1}} > 1$ . We have

$$(\tau \cdot \varphi^{n(\tau)})(L_F(\varphi_q(x))) = (\tau \cdot \varphi^{n(\tau)})(L_F(F_\mu(x)))$$

in  $K_{K_0} \mathbf{B}_{\text{cris}}^+$  and  $h_{\tau^{-1}} > 1$  now implies that  $(\tau \cdot \varphi^{n(\tau)})(L_F(\varphi_q(x)))$  is divisible by  $t_\varpi$  so that by Proposition 2.3, it belongs to  $\text{Fil}^1 \mathbf{B}_{\text{dR}}$ . We can then apply  $L_F^{-1}(T)$  in  $\mathbf{B}_{\text{dR}}$  and get that  $(\tau \cdot \varphi^{n(\tau)})(\varphi_q(x)) = (\tau \cdot \varphi^{n(\tau)})(F_\mu(x))$  in  $\mathbf{B}_{\text{dR}}$ . This equality also holds in  $\tilde{\mathbf{A}}^+$ , so that  $\varphi_q(x) = F_\mu(x)$ .

**Theorem 2.8.** — *There is a Lubin-Tate formal group  $G$  such that  $F_\alpha(T) \in \text{End}(G)$  for all  $\alpha \in O_K$ .*

*Proof.* — By Corollary 2.7, we have  $\varphi_q(x) = F_\mu(x)$ . This implies that  $F_\mu(T) \equiv T^q \pmod{\pi O_K[[T]]}$ . The Weierstrass degree of  $F_\mu(T)$  is  $q^{\text{val}(\mu)}$  so that  $\text{val}(\mu) = 1$  and  $F_\mu(T)$  is a Lubin-Tate power series. By [6], there is a Lubin-Tate formal group  $G$  such that  $F_\mu(T) \in \text{End}(G)$ . Since  $F_\alpha(T)$  commutes with  $F_\mu(T)$ , we also have  $F_\alpha(T) \in \text{End}(G)$  for all  $\alpha \in O_K$ .

**Remark 2.9.** — We have  $\mu = \varpi$  and  $\eta = \chi_\varpi$ . Indeed, the extension  $K_{K_0}/K$  is generated by the torsion points of  $G$ , and is therefore attached to  $\mu$  by local class field theory, so that  $\mu = \varpi$ . This in turn implies that  $\eta = \chi_\varpi$ .

### References

- [1] Pierre Colmez. Espaces de Banach de dimension finie. *J. Inst. Math. Jussieu*, 1(3):331–439, 2002.
- [2] Jean-Marc Fontaine. Le corps des périodes  $p$ -adiques. *Astérisque*, (223):59–111, 1994. With an appendix by Pierre Colmez, *Périodes  $p$ -adiques* (Bures-sur-Yvette, 1988).
- [3] Jean-Marc Fontaine. Représentations  $p$ -adiques semi-stables. *Astérisque*, (223):113–184, 1994.
- [4] Liang-Chung Hsia and Hua-Chieh Li. Ramification filtrations of certain abelian Lie extensions of local fields. *J. Number Theory*, 168:135–153, 2016.
- [5] Jonathan Lubin. Nonarchimedean dynamical systems. *Compositio Math.*, 94(3):321–346, 1994.
- [6] Jonathan Lubin and John Tate. Formal complex multiplication in local fields. *Ann. of Math. (2)*, 81:380–387, 1965.
- [7] Ghassan Sarkis. Height-one commuting power series over  $\mathbb{Z}_p$ . *Bull. Lond. Math. Soc.*, 42(3):381–387, 2010.
- [8] Joel Specter. The crystalline period of a height one  $p$ -adic dynamical system. *Trans. Amer. Math. Soc.*, to appear, 2016.

---

March 30, 2016

LAURENT BERGER, UMPA de l'ENS de Lyon, UMR 5669 du CNRS, IUF  
E-mail : laurent.berger@ens-lyon.fr • Url : perso.ens-lyon.fr/laurent.berger/