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ERRATUM AND ADDENDUM TO "HEIGHTS AND REGULATORS OF NUMBER FIELDS AND ELLIPTIC CURVES"

by

Fabien Pazuki

Abstract. — We give a correct version of the Northcott property for regulators of elliptic curves over number fields.

Résumé. — (*Erratum et addendum de "Hauteurs et régulateurs de corps de nombres et de courbes elliptiques"*) Nous donnons une version corrigée de la propriété de Northcott pour les courbes elliptiques définies sur un corps de nombres.

1. A correction

A square root is missing in the application of the Minkowski inequality on page 59 of [Paz14]. We apologize and correct the statement of Theorem 4.8 page 59 of [Paz14]. We keep the notation of the original article, the regulator is defined using the divisor $L = 3(O)$.

Theorem 1.1. — *Assume the Lang-Silverman Conjecture 4.5 page 58 of* [Paz14]. *Let K be a number field. There exists a quantity $c_{10} = c_{10}(K) > 0$ only depending on K such that for any elliptic curve E defined over K with positive rank m_K,*

$$\mathrm{Reg}(E/K) \geq \Big(\frac{c_{10}}{m_K} \max\{h_{\mathrm{F}^+}(E/K), 1\}\Big)^{m_K}.$$

Thus the set of $\overline{\mathbb{Q}}$-isomorphism classes of elliptic curves defined over a fixed number field K such that $E(K)$ has positive bounded rank m_K and with bounded regulator is finite.

Proof. — Let $L = 3(O)$ and let $\hat{h} = \hat{h}_{E,L}$ be the associated canonical height on E and consider the euclidean space $(E(K) \otimes \mathbb{R}, \hat{h}^{1/2}) \simeq (\mathbb{R}^{m_K}, \hat{h}^{1/2})$. Apply Minkowski's successive minima inequality to the Mordell-Weil lattice $\Lambda_K = E(K)/E(K)_{\mathrm{tors}}$ viewed as a lattice inside this euclidean space,

$$\lambda_1(\Lambda_K) \cdots \lambda_{m_K}(\Lambda_K) \leq m_K^{m_K/2}\, \mathrm{Reg}(E/K)^{1/2}.$$

Now apply m_K times the inequality of Conjecture 4.5 page 58 of [Paz14] to get

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$$\text{Reg}(E/K) \geq \frac{c_4^{m_K} \max\{h_{\mathrm{F}^+}(E/K), 1\}^{m_K}}{m_K^{m_K}}, \tag{1}$$

Finally, if the regulator and the rank is bounded then the height is bounded as soon as $m_K \neq 0$, hence the claimed finiteness. □

It implies the following new statement of Theorem 1.2 page 48 of [Paz14].

Theorem 1.2. — *Assume the Lang-Silverman Conjecture 4.5 page 58 of* [Paz14]*. The set of* $\overline{\mathbb{Q}}$*-isomorphism classes of elliptic curves* E*, defined over a fixed number field* K *with* $E(K)$ *of positive bounded rank* m_K *and bounded regulator is finite.*

2. The number field analogy and a new question

To recover the statement without the boundedness condition, the first idea would be to improve Lemma 4.7 page 58 of [Paz14] (which is still valid and unconditional) to get an inequality of the form $m_K \leq c(K)\, h_{\mathrm{F}^+}(E/K)^{1-\varepsilon}$ for $\varepsilon > 0$ universal and $c(K) > 0$ depending only on K. For $K = \mathbb{Q}$ and under the Generalized Riemann Hypothesis, Mestre has obtained in II.1.2 pages 217-218 of [Mes86] the inequality $m_{\mathbb{Q}} \ll \log N_E / \log\log N_E$ where N_E is the conductor of the elliptic curve E (inequality valid for N_E big enough). It gives hope that a stronger statement than the present Theorem 1.2 could be true.
In order to remove the boundedness condition on the rank, another idea would be to prove an inequality between the regulator and the rank. In view of the number field case, it would play a role similar to Friedman's Theorem 3.4 page 53 of [Paz14]. So we formulate it here as a question.

Question 2.1. — *Let* E *be an elliptic curve defined over a number field* K*. Let* m_K *be the rank of the Mordell-Weil group* $E(K)$ *and let* $\text{Reg}(E/K)$ *be its regulator. Can one find a positive quantity* $c_0(K)$ *depending only on* K *and a strictly increasing function* $f : \mathbb{N} \to \mathbb{R}^+$ *not depending on* E *such that the inequality*

$$\text{Reg}(E/K) \geq c_0(K)\, f(m_K)$$

holds?

Such an inequality would indeed imply that if one fixes K, a bounded regulator would force the rank to be bounded, exactly as in the case of number fields where a bounded regulator implies a bounded degree. This will be the subject of a future work.

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