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Olivier Fouquet

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CONGRUENCES AND THE IWASAWA MAIN CONJECTURE FOR MODULAR FORMS

by

Olivier Fouquet

Abstract. — In the early 2000s, Ralph Greenberg asked whether the Iwasawa Main Conjecture could be proven in a Hida family of nearly-ordinary p-adic eigencuspforms by propagating it from a known case through congruences. Emerton-Pollack-Weston showed that this is indeed possible when the μ-invariant of such a family is trivial. In this article, we show that this is the case without this assumption, and that in fact such a result holds in general over the universal deformation ring of an irreducible residual modular Galois representation.

Résumé. — Au début des années 2000, Ralph Greenberg a demandé si l'on pouvait démontrer la Conjecture Principale de la théorie d'Iwasawa pour une famille de Hida de formes modulaires paraboliques propres quasi-ordinaires en la propageant par congruences à partir d'un cas connu. Emerton-Pollack-Weston ont montré que c'était effectivement possible lorsque l'invariant μ de cette famille est trivial. Dans cet article, nous montrons que c'est le cas sans hypothèse supplémentaire, et que ce résultat demeure en fait vrai sur l'anneau de déformation universelle d'une représentation résiduelle modulaire irréductible.

1. Introduction

1.1. Statement of the theorem. — Let $p > 2$ be an odd prime. Let $f \in S_k(\Gamma_1(N), \varepsilon)$ be a eigencuspform of weight $k \geq 2$. The Iwasawa Main Conjecture for the motive $M(f)$ attached to f is a conjectural description of the p-adic variation of the special values $L_{\{p\}}(f, \psi, r)$ as ψ ranges over finite order character of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ and $1 \leq r \leq k-1$ is an integer. The most general form of this conjecture (and only one known at present to make sense for any f as above) is due to K. Kato and describes this variation in terms of a cohomological invariant called the *fundamental line* and a basis of the fundamental line called the *zeta element* ([21, Conjecture 12.10]).

In the early 2000s, R. Greenberg raised the following question: suppose f corresponds to a classical point in a Hida p-adic family of nearly-ordinary eigencuspforms (in particular, f must itself be nearly-ordinary at p) and suppose that this family contains a classical point

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g for which the Iwasawa Main Conjecture is known to be true, can we then deduce that the Iwasawa Main Conjecture is true for f? The first answer to that question was given in [11], which showed that this was indeed the case provided the residual representation $\bar{\rho}_f$ attached to f satisfied a couple of technical hypotheses and under the assumption that the so-called algebraic and analytic μ-invariants of g were known to be equal to zero. Unfortunately, proving that the algebraic and analytic μ-invariants of an eigencuspform were trivial proved a very hard task (much harder than proving the Iwasawa Main Conjecture). In fact, twenty years after [11], we do not have a single non-tautological criteria allowing us to prove such a vanishing.

In this text, we prove the following theorem.

Theorem 1.1. — *Let Σ be a finite set of primes containing $\{\ell | Np\}$. Let $G_{\mathbb{Q},\Sigma}$ be the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside Σ and ∞. Let k be a finite extension of $\mathbb{F}_p$. Let $\bar{\rho} : G_{\mathbb{Q},\Sigma} \to \mathrm{Gl}_2(k)$ be a Galois representation which satisfies the following hypotheses.*

1. *The $G_{\mathbb{Q}}$-representation $\bar{\rho}$ is isomorphic to $\bar{\rho}_f$ and its image contains a subgroup conjugated to $\mathrm{SL}_2(\mathbb{F}_p)$.*

2. *If $(\bar{\rho}|G_{\mathbb{Q}_p})^{ss} \simeq \bar{\chi}_1 \oplus \bar{\chi}_2$ (that is to say, if the semi-simplification of the residual $G_{\mathbb{Q}_p}$-representation $\bar{\rho}$ is reducible), then $\bar{\chi}_1^{-1}\bar{\chi}_2 \notin \{1, \chi_{\mathrm{cyc}}^{\pm 1}\}$.*

Let $\mathscr{X}_\Sigma(\bar{\rho}) = \operatorname{Spec} R_\Sigma(\bar{\rho})$ be the universal deformation space of $\bar{\rho}$ parametrizing Galois representations $\rho : G_{\mathbb{Q},\Sigma} \to \mathrm{Gl}_2(\mathcal{O})$ with coefficients in a discrete valuation ring $\mathcal{O}$ finite over $\mathbb{Z}_p$ and with residual field k whose residual representations are isomorphic to $\bar{\rho}$.
Then there exists a Zariski-dense open subset $\mathscr{X}_\Sigma^{\mathrm{sm}}(\bar{\rho}) \subset \mathscr{X}_\Sigma(\bar{\rho})$ and a notion of fiber at a modular point of $\mathscr{X}_\Sigma^{\mathrm{sm}}(\bar{\rho})$ such that the following three assertions are equivalent.

i. *The Universal Iwasawa Main Conjecture in families (Conjecture 3.17) is true.*

ii. *The Iwasawa Main Conjecture is true for all modular points in $\mathscr{X}_\Sigma(\bar{\rho})[1/p]$.*

iii. *The Iwasawa Main Conjecture is true for all modular points in $\mathscr{X}_\Sigma^{\mathrm{sm}}(\bar{\rho})[1/p]$.*

iv. *The Iwasawa Main Conjecture is true for the points in a single fiber of a classical point of $\mathscr{X}_\Sigma^{\mathrm{sm}}(\bar{\rho})[1/p]$.*

By definition, modular points of $\mathscr{X}_\Sigma(\bar{\rho})[1/p]$ correspond to Galois representations attached to classical eigencuspforms congruent to f and the fiber at a modular point of $\mathscr{X}_\Sigma^{\mathrm{sm}}(\bar{\rho})[1/p]$ is a finite set of classical eigencuspforms, so Theorem 1.1 establishes that the Iwasawa Main Conjecture is true for f provided it is true for a well-chosen finite set of eigencuspforms congruent to f. Compared to [11], we do not require vanishing of μ-invariants and we do not require f to be ordinary at p (in fact, we impose no hypothesis on the eigencuspform f, only on its residual representation $\bar{\rho}$). In particular, we note that Theorem 1.1 appears to be new even if we add the supplementary hypothesis that $\rho_f|G_{\mathbb{Q}_p}$ is ordinary and/or crystalline.

Acknowledgments. — The material presented in this text has been inspired by numerous conversations with R. Pollack and X. Wan. It is a pleasure to thank the referee(s) for several improvements.

1.2. Outline of the proof. — The proof of Theorem 1.1 is by p-adic variation. Its main steps are as follows. We first recall that $\mathscr{X}_\Sigma(\bar{\rho})$ is a complete intersection scheme of relative dimension 3 over $\mathbb{Z}_p$ and flat over $\operatorname{Spec}\boldsymbol{\Lambda}$ where $\boldsymbol{\Lambda}$ is power-series ring $\mathcal{O}[\![X_1,X_2,X_3]\!]$. We define $\mathscr{X}_\Sigma^{\mathrm{sm}}(\bar{\rho})$ using deformation theory and show that points in $\mathscr{X}_\Sigma^{\mathrm{sm}}(\bar{\rho})$ are smooth over $\operatorname{Spec}\boldsymbol{\Lambda}$. Then, we construct a fundamental line Δ_Σ on $\mathscr{X}_\Sigma(\bar{\rho})$, which is a free module of rank 1 over the power-series ring $\boldsymbol{\Lambda}$, and recall the existence of a universal zeta morphism

$$\mathbf{z}_\Sigma : \Delta_\Sigma \otimes_{\boldsymbol{\Lambda}} \operatorname{Frac}(\boldsymbol{\Lambda}) \xrightarrow{\sim} \operatorname{Frac}(\boldsymbol{\Lambda})$$

constructed in [25]. The universal Iwasawa Main Conjecture is then the statement that this universal zeta morphism induces

$$\mathbf{z}_\Sigma : \Delta_\Sigma \xrightarrow{\sim} \boldsymbol{\Lambda}.$$

We conclude by showing that this universal Iwasawa Main Conjecture is implied by the Iwasawa Main Conjecture at the fiber of an arbitrary single modular point in $\mathscr{X}_\Sigma^{\mathrm{sm}}(\bar{\rho})$ and that it implies the Iwasawa Main Conjecture at all modular points of $\mathscr{X}_\Sigma(\bar{\rho})$. This establishes the theorem.

Iwasawa theory started when K. Iwasawa accounted for the growth of the cardinal of the p-part of the ideal class group of $\mathbb{Z}[\zeta_{p^n}]$ as n goes to infinity by relating the class groupe $\operatorname{Cl}(\mathbb{Z}[\zeta_{p^n}])$ to the direct limit of these class groups, or in modern language by relating the specialization at a classical point of Λ-adic Selmer module to the Selmer module at this classical point. After [23, 16], this kind of result has been known as a control theorem (because it establishes the large object — the Λ-adic Selmer module — *controls* each Selmer modules over small objects). An important step in the chain reductions proving the theorem is Proposition 3.15, which shows that fundamental lines and zeta elements are as well-behaved as they can possibly be in the universal deformation. This proposition is thus an optimal and maximally general version of the control theorem for p-adic families of eigencuspforms.

2. Universal deformation rings

As in the introduction, we consider Σ a finite set of primes containing $\{\ell|Np\}$. Let $G_{\mathbb{Q},\Sigma}$ be the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside Σ and ∞. Let $E/\mathbb{Q}_p$ be a finite extension with unit ball $\mathcal{O}$ and residual field k. Let $\bar{\rho}: G_{\mathbb{Q},\Sigma} \to \mathrm{Gl}_2(k)$ be a Galois representation which is the residual representation attached to an eigencuspform $f \in S_k(\Gamma_1(N),\varepsilon)$ wit $k \geq 2$. We assume in addition that $\bar{\rho}$ satisfies hypotheses 1 and 2 of Theorem 1.1.

Let $D_{\bar{\rho}}$ be the deformation functor from the category of finite, local, $\mathcal{O}$-algebras with residue field k to the category of sets which attaches to A the set of isomorphism casses of pairs $\{(T,\rho,A),\iota)\}$ such that (T,ρ,A) is a $G_{\mathbb{Q},\Sigma}$-representation and $\iota : \rho\otimes_A k \simeq \bar{\rho}$ is an isomorphism of $k[G_{\mathbb{Q},\Sigma}]$-module. As $\bar{\rho}$ is absolutely irreducible, $D_{\bar{\rho}}$ is representable by a complete, local, noetherian ring $R_\Sigma(\bar{\rho})$. We denote by $(T_\Sigma,\rho_\Sigma,R_\Sigma(\bar{\rho}))$ the corresponding universal $G_{\mathbb{Q},\Sigma}$-representation and by $\mathscr{X}_\Sigma(\bar{\rho}) = \operatorname{Spec} R_\Sigma(\bar{\rho})[1/p]$ the universal deformation space.

By definition, if A is a complete, local, noetherian, flat $\mathcal{O}$-algebra with residual field k, x is an A-valued point of $\mathscr{X}_\Sigma(\bar{\rho})$ if and only if there exists a Galois representation

$$\rho_x : G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{Gl}_2(A)$$

which satisfies

$$\operatorname{tr}(\rho(\operatorname{Fr}(\ell))) \bmod \mathfrak{m}_A = \operatorname{tr}(\bar{\rho}(\operatorname{Fr}(\ell))), \quad \det(\rho(\operatorname{Fr}(\ell))) \bmod \mathfrak{m}_A = \det(\bar{\rho}(\operatorname{Fr}(\ell)))$$

for all $\ell \notin \Sigma$. We say that an $\mathcal{O}$-valued point $x \in \mathscr{X}_\Sigma(\bar\rho)$ is modular or classical if there exists an eigencuspform

$$g_x(z) = \sum_{n=1}^{\infty} a_\ell(g_x) q^n$$

of weight $k \geq 2$ such that $\rho_{g_x} \simeq \rho_\Sigma \otimes_{R_\Sigma(\bar\rho),x} \mathcal{O}$. If x and y are modular points, $a_\ell(g_x) \equiv a_\ell(g_y)$ for all $\ell \notin \Sigma$. In other words, the eigencuspforms g_x and g_y are congruent. As ρ_f is itself a modular point of $\mathscr{X}_\Sigma(\bar\rho)$ by construction, any eigencuspform attached to a modular point of $\mathscr{X}_\Sigma(\bar\rho)$ is congruent to f.
We record the following theorem, which is presumably well-known.

Theorem 2.1. — *The universal deformation ring $R_\Sigma(\bar\rho)$ is a flat $\mathcal{O}$-algebra which is a reduced, complete intersection ring of Krull dimension 4. There exists an open subset $\mathscr{X}^{\mathrm{sm}} \subset \mathscr{X}_\Sigma(\bar\rho)$ dense for the Zariski and the adic topology containing classical points on each irreducible component of $\mathscr{X}_\Sigma(\bar\rho)$ such that the map*

$$\begin{array}{c}\mathscr{X}_\Sigma(\bar\rho)\\ \big\downarrow\\ \operatorname{Spec}\mathcal{O}\end{array}$$

is formally smooth at $x \in \mathscr{X}^{\mathrm{sm}}$.

Proof. — We give a brief proof.
Let k be the Serre-weight of $\bar\rho$. According to [10], there exists a point $x \in \mathscr{X}_\Sigma(\bar\rho)[1/p]$ attached to a classical eigencuspform f of weight k. Denote by $\chi : G_\mathbb{Q} \to \mathcal{O}^\times$ the character $\det\rho_f$. Let $R_\Sigma^\chi(\bar\rho)$ be the deformation ring of $\bar\rho$ parametrizing deformations ρ of $\bar\rho$ with coefficients in complete noetherian local $\mathcal{O}$-algebras, unramified outside Σ and such that $\det\rho$ is equal to χ (in particular, ρ_f corresponds to a point in $R_\Sigma^\chi(\bar\rho)$). Let ρ be an arbitrary deformation of $\bar\rho$ corresponding to a point of $R_\Sigma(\bar\rho)$ with coefficients in $\mathcal{O}^\times$. Then $(\det\rho)^{-1}\chi$ has values in $\mathcal{O}^\times$ and $(\det\rho)^{-1}\chi \equiv (\det\bar\rho)^{-1}(\det\bar\rho_f)^{-1} \equiv 1 \bmod \varpi$. So $(\det\bar\rho)^{-1}\chi$ has values in $1+\varpi\mathcal{O}$ and corresponds to a point of the universal deformation ring $R_\Sigma(\mathbb{1})$ parametrizing deformations $\psi : G_{\mathbb{Q},\Sigma} \to \mathcal{O}^\times$ of the trivial character $\mathbb{1}$ of $G_{\mathbb{Q},\Sigma}$. As p is odd, the multiplicative group $1+\varpi\mathcal{O}$ is uniquely 2-divisible so characters with values in $1+\varpi\mathcal{O}$ admit canonical square roots. Let ψ_ρ be the canonical square root of $(\det\rho)^{-1}\chi$. Then $\rho\otimes\psi_\rho \equiv \bar\rho \bmod \varpi$ and $\det(\rho\otimes\psi_\rho) = \chi$ so $\rho\otimes\psi_\rho$ corresponds to a point of $R_\Sigma^\chi(\bar\rho)$. The map $\rho \mapsto \rho\otimes\psi_\rho$ thus induces an isomorphism $R_\Sigma(\bar\rho) \simeq R_\Sigma^\chi(\bar\rho)\hat\otimes_\mathcal{O} R_\Sigma(\mathbb{1})$. As $R_\Sigma(\mathbb{1})$ is isomorphic to a power-series ring in one-variable over a complete intersection $\mathcal{O}$-algebra of relative dimension zero, $R_\Sigma(\bar\rho)$ is a flat $\mathcal{O}$-algebra which is a complete intersection ring of relative dimension 3 if and only if $R_\Sigma^\chi(\bar\rho)$ is a flat $\mathcal{O}$-algebra which is a complete intersection ring of relative dimension 2. This we now show.
Assume first that $\bar\rho|G_{\mathbb{Q}_p}$ is reducible. Twisting by a character if necessary, we may then assume that $(\bar\rho|G_{\mathbb{Q}_p})^{ss} = \bar\chi_1 \oplus \bar\chi_2$ with $\bar\chi_1(I_p) = \{1\}$. Let $R_\Sigma^{\mathrm{ord},\chi}(\bar\rho)$ be the quotient of $R_\Sigma^\chi(\bar\rho)$ parametrizing deformations ρ which in addition to being points of $R_\Sigma^\chi(\bar\rho)$ are such that there exists a short exact sequence of non-zero $G_{\mathbb{Q}_p}$-representations

$$0 \longrightarrow \chi_1 \longrightarrow \rho|G_{\mathbb{Q}_p} \longrightarrow \chi_2 \longrightarrow 0$$

with $\chi_1(I_p) = \{1\}$. According to the main results of [8, 15] (following [38, 36]), the ring $R_\Sigma^{\mathrm{ord},\chi}(\bar\rho)$ is isomorphic to a suitable Hecke algebra and hence flat of relative dimension 0 over

$\mathcal{O}$. Suppose now that $\bar\rho|G_{\mathbb{Q}_p}$ is irreducible. Without loss of generality, we may then assume that the eigencuspform f above is crystalline at p of weight $2 \leq k \leq p$ (see [33]). Denote by $R_\Sigma^{\text{crys},\chi}(\bar\rho)$ the quotient of $R_\Sigma^\chi(\bar\rho)$ parametrizing deformations ρ which in addition to being points of $R_\Sigma^\chi(\bar\rho)$ are such that $\rho|G_{\mathbb{Q}_p}$ is crystalline of weight $2 \leq k \leq p$. By the modularity result of [9], the ring $R_\Sigma^{\text{crys},\chi}(\bar\rho)$ is isomorphic to a suitable Hecke algebra and hence flat of relative dimension 0 over $\mathcal{O}$. We have thus seen that the ring $R_\Sigma^{*,\chi}(\bar\rho)$ is flat of relative dimension 0 over $\mathcal{O}$ both when $\bar\rho|G_{\mathbb{Q}_p}$ is reducible and $* = \text{ord}$ and when $\bar\rho|G_{\mathbb{Q}_p}$ is irreducible and $* = \text{crys}$.

Next we describe the kernel of the map $R_\Sigma^\chi(\bar\rho) \to R_\Sigma^{*,\chi}(\bar\rho)$ when $\bar\rho|G_{\mathbb{Q}_p}$ is reducible and $* = \text{ord}$ and when $\bar\rho|G_{\mathbb{Q}_p}$ is irreducible and $* = \text{crys}$. Assume first that $\bar\rho|G_{\mathbb{Q}_p}$ is reducible. Under our assumptions, the universal framed ordinary deformation ring $R^{\square,\text{ord}}(\bar\rho|G_{\mathbb{Q}_p})$ is a regular ring of relative dimension 3 (see [34, Section 2.2] and especially Lemma 2.2 there for the notion of framed ordinary deformation ring and a proof of the previous statement). According to [2, Corollary 7.4], $R^{\square,\text{ord}}(\bar\rho|G_{\mathbb{Q}_p})$ is a quotient of $R^{\square}(\bar\rho|G_{\mathbb{Q}_p})$ by a length two regular sequence. As a quotient of a ring by a regular sequence is regular (if and) only if the ring itself was regular, this entails that $R^{\square}(\bar\rho|G_{\mathbb{Q}_p})$ is a regular ring of relative dimension 5 over $\mathcal{O}$ and that kernel of the map $R^{\square}(\bar\rho|G_{\mathbb{Q}_p}) \to R^{\square,\text{ord}}(\bar\rho|G_{\mathbb{Q}_p})$ is generated by a subset of a system of parameters of cardinal 2. Now we assume that $\bar\rho|G_{\mathbb{Q}_p}$ is irreducible. Then [9, Proposition 2.2 and Corollary 2.3] (and previous results of [30]) imply that the universal deformation ring $R(\bar\rho|G_{\mathbb{Q}_p})$ is a regular ring of relative dimension 5 and that $R^{\text{crys}}(\bar\rho|G_{\mathbb{Q}_p})$ is a regular quotient of relative dimension 2. Let $R^\chi(\bar\rho|G_{\mathbb{Q}_p})$ and $R^{\text{crys},\chi}(\bar\rho|G_{\mathbb{Q}_p})$ respectively denote the quotient of the universal deformation ring parametrizing deformations ρ of $\bar\rho|G_{\mathbb{Q}_p}$ with coefficients in complete noetherian local $\mathcal{O}$-algebras and such that $\det\rho$ is equal to χ and the quotient parametrizing such deformations which are in addition crystalline of weight $2 \leq k \leq p-1$. By local class field theory, $R^\chi(\bar\rho|G_{\mathbb{Q}_p})$ is then a regular ring of relative dimension 3 and $R^{\text{crys},\chi}(\bar\rho|G_{\mathbb{Q}_p})$ is a regular ring of relative dimension 1. The kernel of $R^\chi(\bar\rho|G_{\mathbb{Q}_p}) \to R^{\text{crys},\chi}(\bar\rho|G_{\mathbb{Q}_p})$ is thus generated by a regular sequence of length 2.

Denoting by $R^\chi(\bar\rho|G_{\mathbb{Q}_p})$ and $R^{*,\chi}(\bar\rho|G_{\mathbb{Q}_p})$ the quotient of the framed or usual universal deformation ring parametrizing deformations ρ of $\bar\rho|G_{\mathbb{Q}_p}$ with coefficients in complete noetherian local $\mathcal{O}$-algebras and such that $\det\rho$ is equal to χ (resp. which in addition are of type $* \in \{\text{ord}, \text{crys}\}$), we consequently see that in both the reducible and irreducible case, there is a commutative diagram

$$\begin{array}{ccc} R^\chi(\bar\rho|G_{\mathbb{Q}_p}) & \longrightarrow & R^{*,\chi}(\bar\rho|G_{\mathbb{Q}_p}) \\ \downarrow & & \downarrow \\ R_\Sigma^\chi(\bar\rho) & \longrightarrow & R_\Sigma^{*,\chi}(\bar\rho) \end{array}$$

where the vertical map are induced by restriction from $G_{\mathbb{Q},\Sigma}$ to $G_{\mathbb{Q}_p}$ and where the kernel of the upper horizontal arrow is generated by a regular sequence of length 2. This implies that $R_\Sigma^{*,\chi}(\bar\rho)$ is a quotient of $R_\Sigma^\chi(\bar\rho)$ by a an ideal generated by at most two elements, and thus that $R_\Sigma^{*,\chi}(\bar\rho)$ is a quotient of $R_\Sigma(\bar\rho)$ by an ideal (x_1, x_2, x_3) generated by at most three elements. This entails in particular that $R_\Sigma(\bar\rho)$ is of Krull dimension at most 4.

According to [3, Lemma 4.3], the ring $R_\Sigma(\bar\rho)$ is of dimension at least 4 so we conclude that it is flat of relative dimension 3 over $\mathcal{O}$. The same lemma shows that the zero-dimensional ring

$R^{*,\chi}_{\Sigma}(\bar{\rho})/(\varpi)$ admits a presentation

$$R^{*,\chi}_{\Sigma}(\bar{\rho})/(\varpi) \simeq \mathcal{O}[\![X_1,\ldots,X_n]\!]/(\varpi, x_1, x_2, x_3, y_4, \ldots, y_n)$$

As $R^{*,\chi}_{\Sigma}(\bar{\rho})/(\varpi)$ is of Krull dimension zero, this implies that $(\varpi, x_1, x_2, x_3, y_4, \ldots, y_n)$ is a regular sequence in $\mathcal{O}[\![X_1,\ldots,X_n]\!]$. In particular, $R^{*,\chi}_{\Sigma}(\bar{\rho})/(\varpi)$, $R^{*,\chi}_{\Sigma}(\bar{\rho})$ and $R_{\Sigma}(\bar{\rho})$ are complete intersection rings of dimension 0, 1 and 4 respectively. It follows in addition that (x_1, x_2, x_3) is a regular sequence in $R_{\Sigma}(\bar{\rho})$.
Let (x_2, x_3) be the sub-regular sequence extracted from (x_1, x_2, x_3) above and let $\mathbf{\Lambda}$ be the power-series ring $\mathcal{O}[\![X_2, X_3]\!]$. We consider the commutative diagram

$$\begin{array}{ccc} \mathbf{\Lambda} & \xrightarrow{X_i \mapsto x_i} & R^{\chi}_{\Sigma}(\bar{\rho}) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathcal{O} & \longrightarrow & R^{*,\chi}_{\Sigma}(\bar{\rho}) \end{array}$$

where the vertical maps π and π' are the quotient maps modulo (X_2, X_3) and (x_2, x_3) respectively. Let $\mathfrak{p} \in \operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho})$ be a point above $\pi^* : \operatorname{Spec}\mathcal{O} \to \operatorname{Spec}\mathbf{\Lambda}$. As $R^{\chi}_{\Sigma}(\bar{\rho})_{\mathfrak{p}}$ is flat over $R^{\chi}_{\Sigma}(\bar{\rho})$, the regular sequence (x_2, x_3) of $R^{\chi}_{\Sigma}(\bar{\rho})$ remains a regular sequence of $R^{\chi}_{\Sigma}(\bar{\rho})_{\mathfrak{p}}$. The quotient $R^{\chi}_{\Sigma}(\bar{\rho})_{\mathfrak{p}}/(x_2, x_3)$ is by construction the localization at a minimal prime ideal of the reduced, classical Hecke algebra of appropriate level and weight after extension of scalars to the fraction field E of $\mathcal{O}$. Hence, it is a separable field extension and $\operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho})_{\mathfrak{p}}/(x_2, x_3) \to \operatorname{Spec} E$ is an étale morphism. Consequently, $\operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho}) \to \operatorname{Spec}\mathbf{\Lambda}$ is unramified at $\mathfrak{p}$. As it is also finite and flat by the first part of the proof above, the locus $\mathscr{X}^{\mathrm{sm}}$ of étale points of $\operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho}) \to \operatorname{Spec}\mathbf{\Lambda}$ contains $\mathfrak{p}$ and is thus non-empty. Let $U \subset \operatorname{Spec}\mathbf{\Lambda}$ be the complement of the support of $\Omega^1_{R^{\chi}_{\Sigma}(\bar{\rho})/\mathbf{\Lambda}}$ regarded as $\mathbf{\Lambda}$-module. By the above, U is non-empty, formally smooth over $\operatorname{Spec}\mathcal{O}$ and $\mathscr{X}^{\mathrm{sm}}$ is formally smooth over U. Hence $\mathscr{X}^{\mathrm{sm}}$ is formally smooth over $\operatorname{Spec}\mathcal{O}$. By construction, U is open and non-empty, hence Zariski-dense in $\operatorname{Spec}\mathbf{\Lambda}$. So $\operatorname{Spec}\mathbf{\Lambda}\backslash U$ is of codimension at least 1. As $\operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho}) \to \operatorname{Spec}\mathbf{\Lambda}$ is finite, $\operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho})\backslash\mathscr{X}^{\mathrm{sm}}$ is also of codimension at least 1. As $R^{\chi}_{\Sigma}(\bar{\rho})$ is Cohen–Macaulay, it is equidimensional. Hence $\mathscr{X}^{\mathrm{sm}}$ is Zariski-dense in each irreducible component. Let $\{\mathfrak{a}_i \,|\, i \in I\} \subset \operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho})$ be the finite set of minimal prime ideals of $R^{\chi}_{\Sigma}(\bar{\rho})$ and write $\mathscr{X}_i \overset{\text{def}}{=} \operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho})/\mathfrak{a}_i$. Then the sets $\mathscr{X}^{\mathrm{sm}} \cap \mathscr{X}_i$ are non-empty and disjoint. As $\mathscr{X}^{\mathrm{sm}} \cap \mathscr{X}_i$ is étale over U, its generic degree is equal to its degree at $\mathfrak{p}$. Hence, each $\mathscr{X}_i$ contains a point $\mathfrak{p}_i \in \mathscr{X}^{\mathrm{sm}} \cap \mathscr{X}_i$ above π^* and the $\mathfrak{p}_i$ are pairwise distinct. The isomorphism between suitable deformation rings $R^{\chi,*}_{\Sigma}(\bar{\rho})$ and Hecke algebras moreover shows that each irreducible component of $R_{\Sigma}(\bar{\rho})$ contains points attached to eigencuspforms.
As $R^{\chi}_{\Sigma}(\bar{\rho})$ is a Cohen–Macaulay ring, the set $\{\mathfrak{a}_i | i \in I\}$ is also the set of associated primes of $R^{\chi}_{\Sigma}(\bar{\rho})$. By the properties of $\mathscr{X}^{\mathrm{sm}}$ just established, for each $i \in I$, there exists an element $a_i \in R^{\chi}_{\Sigma}(\bar{\rho})$ such that $a_j \mod \mathfrak{a}_i$ vanishes if and only if $i \neq j$ and such that $\operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho})_{a_i}$ is smooth over E. Then $a = \sum_{i\in I} a_i$ does not belong to any minimal prime $\mathfrak{a}_i$, hence does not belong to an associated prime of $R^{\chi}_{\Sigma}(\bar{\rho})$. Hence a is not a zero-divisor and there is an embedding $R^{\chi}_{\Sigma}(\bar{\rho}) \hookrightarrow R^{\chi}_{\Sigma}(\bar{\rho})_a$. As $\operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho})_a \simeq \coprod_{i\in I} \operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho})_{a_i}$ and as $\operatorname{Spec} R^{\chi}_{\Sigma}(\bar{\rho})_{a_i}$ is smooth, $R^{\chi}_{\Sigma}(\bar{\rho})$ is reduced. The same must then be true of $R_{\Sigma}(\bar{\rho})$ and we have finally established all assertions of the theorem. ☐

If $x \in \mathscr{X}_{\Sigma}^{\mathrm{sm}}(\bar{\rho})$ is a classical point, we call the fiber of x the finite set of points x_i such that the x_i are above the same point of $\operatorname{Spec} \mathcal{O}[\![X_1, X_2, X_3]\!]$ for the structure morphism of the proof above (in particular, if x is nearly-ordinary, then all points in the fiber are nearly-ordinary and if x is crystalline and short, then all points in the fiber are crystalline and short - though we will not make use of this particular fact).

3. The Iwasawa Main Conjecture for modular forms: variations on a theme

3.1. Modular motives. — As in the introduction, we consider

$$f(z) = \sum_{n=1}^{\infty} a_n(f) q^n \in S_k(\Gamma_1(N), \varepsilon)$$

an eigencuspform of weight $k \geq 2$, level $\Gamma_1(N)$ and nebentypus ε with coefficients in a number field $F \subset \mathbb{C}$.
To f and $1 \leq r \leq k-1$ is attached a Grothendieck motive $M(f)(r)$ pure of weight $k-1$ over $\mathbb{Q}$ and with coefficients in F ([32]). We fix $\mathfrak{p}|p$ a prime ideal of $\mathcal{O}_F$ over p and write E for $F_{\mathfrak{p}}$ and $\mathcal{O}$ for the unit ball of E. For $* \in \{\mathbb{C}, \mathrm{dR}, \mathfrak{p}\}$, We write $V(r)_*$ for the corresponding Betti, de Rham or $\mathfrak{p}$-adic étale realization of $M(f)(r)$. We recall that $V(r)_{\mathbb{C}}$ and $V(r)_{\mathfrak{p}}$ are endowed with an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and that Artin comparison theorem

$$V(r)_{\mathbb{C}} \otimes_F E \overset{\mathrm{can}}{\simeq} V(r)_{\mathfrak{p}}$$

is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant for this action. We denote by $(-)^+$ the functor $H^0(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), -)$.
Let $\zeta_{p^{n+1}}$ be a primitive root of unity of order p^{n+1} and let $\mathbb{Q}_n/\mathbb{Q}$ be the subfield of $\mathbb{Q}(\zeta_{p^{n+1}})$ with Galois group G_n over $\mathbb{Q}$ equal to $\mathbb{Z}/p^n\mathbb{Z}$. If $\chi \in \widehat{G}_n$ is a character of G_n, we denote by F_χ the extension of $\mathbb{Q}$ generated by F and the image of χ. We write E_χ and $\mathcal{O}_\chi$ for a choice of corresponding finite extension of E. We consider $h^0(\operatorname{Spec} \mathbb{Q}_n)$ the Artin motive attached to the regular representation of G_n viewed as a pure motive over $\mathbb{Q}$ with coefficients in F_χ and denote by $h^0(\operatorname{Spec} \mathbb{Q}_n)_\chi$ the direct summand of $h^0(\operatorname{Spec} \mathbb{Q}_n)$ on which G_n acts through χ. Let $M(f \otimes \chi)(r)$ be the pure Grothendieck motive $M(f)(r) \times h^0(\operatorname{Spec} \mathbb{Q}_n)_\chi$ with coefficients in F_χ. We write $V(r)_{\chi,*}$ for the corresponding $*$-realization.
For S a set of finite primes containing $\{p\}$, the S-partial L-function $L_S(f^*, \chi, s)$ is the holomorphic complex function satisfying

$$L_S(f^*, \chi, s) \overset{\mathrm{def}}{=} \prod_{\ell \notin S} \frac{1}{1 - \overline{a_\ell} \chi(\mathrm{Fr}(\ell)) \ell^{-s} + \overline{\varepsilon(\ell)} \chi(\mathrm{Fr}(\ell)) \ell^{k-1-2s}}$$

for all $s \in \mathbb{C}$ with $\Re s >> 0$ (here $\bar{\cdot}$ denotes complex conjugation).
The Betti-de Rham comparison isomorphism induces a complex period map

$$\mathrm{per}_{\mathbb{C}} : \mathrm{Fil}^0 V_{\chi,\mathrm{dR}}(r) \otimes_{F_\chi} \mathbb{C} \longrightarrow V_{\chi,\mathbb{C}}(r-1)^+ \otimes_{F_\chi} \mathbb{C}. \tag{1}$$

which is an isomorphism as $1 \leq r \leq k-1$ ([7]). The composition of localization at p with the dual exponential map $\exp^*$ of [1]

$$\exp^* : H^1(G_{\mathbb{Q}_p(\zeta_{p^n})}, V(r)_{\mathfrak{p}}) \longrightarrow D^0_{\mathrm{dR}}(V(r)_{\mathfrak{p}})$$

induces an inverse p-adic period map of E_χ-vector spaces

$$\mathrm{per}_p^{-1} : H^1_{\mathrm{et}}(\mathbb{Z}[1/S], V(r)_{\chi,\mathfrak{p}}) \longrightarrow \mathrm{Fil}^0 V(r)_{\chi,\mathrm{dR}} \otimes_F E_\chi \tag{2}$$

which is equivariant under the action of G_n on both sides. According to [31] and [21, Theorems 12.4 and 12.5], for all $n \in \mathbb{N}$ and all $\chi \in \widehat{G}_n$ except possibly finitely many, $H^1_{\mathrm{et}}(\mathbb{Z}[1/S], V(r)_{\chi,\mathfrak{p}})$ is an E_χ-vector space of dimension 1 and (2) is an isomorphism. We say that $M(f\otimes\chi)(r)$ is strictly critical when this holds ([19, Section 3.2.6]).

3.2. The Tamagawa Number Conjecture. —

3.2.1. *The motivic fundamental line.* — Suppose that $M(f\otimes\chi)(r)$ is strictly critical. The determinant functor applied to the p-adic period map (2) then yields an isomorphism of free E_χ-vector spaces of rank 1

$$\begin{array}{c}\mathrm{Det}_{E_\chi} H^1_{\mathrm{et}}(\mathbb{Z}[1/S], V(r)_{\chi,p}) \\ {\scriptstyle\mathrm{can}}\ \simeq \Big\downarrow \mathrm{per}_p^{-1} \\ \mathrm{Det}_{E_\chi} \mathrm{Fil}^0 V(r)_{\chi,\mathrm{dR}} \otimes_F E_\chi.\end{array}$$

Taking tensor product with the determinant of $V(r-1)^+_{\chi,\mathfrak{p}}$ (recall that $(-)^+$ sends a module to its invariant submodule under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$) yields an identification

$$\begin{array}{c}\mathrm{Det}_{E_\chi} H^1_{\mathrm{et}}(\mathbb{Z}[1/S], V(r)_{\chi,\mathfrak{p}}) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V(r-1)^+_{\chi,\mathfrak{p}} \\ {\scriptstyle\mathrm{can}}\ \simeq \Big\downarrow \mathrm{per}_p^{-1} \\ \mathrm{Det}_{E_\chi}\left(\mathrm{Fil}^0 V(r)_{\chi,\mathrm{dR}} \otimes_F E_\chi\right) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V(r-1)^+_{\chi,\mathfrak{p}}.\end{array} \tag{3}$$

Similarly, the determinant functor applied to the complex period map (1) induces an identification

$$\begin{array}{c}\left[\mathrm{Det}_{\mathbb{C}} \mathrm{Fil}^0 V(r)_{\chi,\mathrm{dR}} \otimes_{F_\chi} \mathbb{C}\right] \otimes \left[\mathrm{Det}^{-1}_{\mathbb{C}} V(r-1)^+_{\chi,\mathbb{C}} \otimes_{F_\chi} \mathbb{C}\right] \\ {\scriptstyle\mathrm{can}}\ \simeq \Big\downarrow \mathrm{per}_{\mathbb{C}} \\ \mathbb{C}.\end{array} \tag{4}$$

Definition 3.1. — The *motivic fundamental line* $\left(\Delta_{\mathrm{mot}}(M(f\otimes\chi)(r)), \mathrm{per}_p, \mathrm{per}_{\mathbb{C}}\right)$ of the strictly critical motive $M(f\otimes\chi)(r)$ is the one-dimensional F_χ-vector space

$$\Delta_{\mathrm{mot}}(M(f\otimes\chi)(r)) \stackrel{\mathrm{def}}{=} \mathrm{Det}_{F_\chi} \mathrm{Fil}^0 V(r)_{\chi,\mathrm{dR}} \otimes_{F_\chi} \mathrm{Det}^{-1}_{F_\chi} V(r-1)^+_{\chi,\mathbb{C}} \tag{5}$$

together with the two isomorphisms

$$\mathrm{per}_p : \Delta_{\mathrm{mot}}(M(f\otimes\chi)(r)) \otimes_{F_\chi} E_\chi \xrightarrow{\sim} \mathrm{Det}_{E_\chi} H^1_{\mathrm{et}}(\mathbb{Z}[1/S], V(r)_{\chi,\mathfrak{p}}) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V(r-1)^+_{\chi,\mathfrak{p}}$$
$$\mathrm{per}_{\mathbb{C}} : \Delta_{\mathrm{mot}}(M(f\otimes\chi)(r)) \otimes_{F_\chi} \mathbb{C} \xrightarrow{\sim} \mathbb{C}.$$

Note that though this is not apparent in the notations, the isomorphism per_p and hence the motivic fundamental line depend on the choice of S. Note also that motivic fundamental line of $M(f\otimes\chi)(r)$ is an F_χ-rational subspace both of the target of (3) and of the source of (4). Consequently, to any element $\mathbf{z}$ of the source of (3) whose image through per_p^{-1} lands in the motivic fundamental line $\Delta_{\mathrm{mot}}(M(f\otimes\chi)(r))$ is attached a complex number $\mathrm{per}_{\mathbb{C}}(\mathrm{per}_p^{-1}(\mathbf{z})\otimes 1)$. Let $\mathrm{per}_{\mathbb{C}}(\mathrm{per}_p^{-1}(-)\otimes 1)$ be the map defined by this composition on the sub-F_χ-subspace $\Delta_{\mathrm{mot}}(M(f\otimes\chi)(r))_{\mathfrak{p}}$ equal to the image of $\Delta_{\mathrm{mot}}(M(f\otimes\chi)(r))$ through per_p.

3.2.2. *Zeta morphisms.* — We still assume that $M(f)(r)$ is strictly critical. In that case, let

$$Z : V(r-1)^{+}_{\chi,\mathfrak{p}} \xrightarrow{\sim} H^1_{\mathrm{et}}(\mathbb{Z}[1/S], V(r)_{\chi,\mathfrak{p}})$$

be an isomorphism between the 1-dimensional E_χ-vector spaces $V(r-1)^{+}_{\chi,\mathfrak{p}}$ and $H^1_{\mathrm{et}}(\mathbb{Z}[1/S], V(r)_{\chi,\mathfrak{p}})$. Applying the functor Det then yields an identification

$$\mathrm{Det}_{E_\chi}(Z) : \mathrm{Det}_{E_\chi} H^1_{\mathrm{et}}(\mathbb{Z}[1/S], V(r)_{\chi,\mathfrak{p}}) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V(r-1)^{+}_{\chi,\mathfrak{p}} \simeq E_\chi.$$

Let $\Delta_{F,Z}(M(f\otimes\chi)(r))$ be the F_χ-vector space pre-image of $F_\chi \subset E_\chi$ through this isomorphism.

Definition 3.2. — A morphism

$$Z : V(r-1)^{+}_{\chi,\mathfrak{p}} \xrightarrow{\sim} H^1_{\mathrm{et}}(\mathbb{Z}[1/S], V(r)_{\chi,\mathfrak{p}})$$

is *motivic* if $\Delta_{F,Z}(M(f\otimes\chi)(r))$ is equal to the image $\Delta_{\mathrm{mot}}(M(f\otimes\chi)(r))_{\mathfrak{p}}$ of $\Delta_{\mathrm{mot}}(M(f\otimes\chi)(r))$ through per_p. If a morphism Z is motivic, we say it is the *S-partial zeta morphism* of $M(f\otimes\chi)(r)$ if $\mathrm{per}_{\mathbb{C}}(\mathrm{per}_p^{-1}(\mathrm{Det}_{E_\chi}(Z)^{-1}(1))\otimes 1)$ is equal to $L_S(f^*, \chi^{-1}, 1-r) \in \mathbb{C}$.

Note that the composition $\mathrm{per}_{\mathbb{C}}(\mathrm{per}_p^{-1}(\mathrm{Det}_{E_\chi}(Z)^{-1}(1))\otimes 1)$ makes sense as the pre-image of $\mathrm{Det}_{E_\chi}(Z)^{-1}(1)$ through per_p belongs to $\Delta_{\mathrm{mot}}(M(f\otimes\chi)(r))$ if (and only if) Z is motivic. An S-partial zeta morphism is unique, if it exists.
Taking the determinant functor of a zeta morphism induces a trivialization

$$Z : \mathrm{Det}^{-1}_{E_\chi} \mathrm{R}\Gamma_{\mathrm{et}}\,(\mathbb{Z}[1/S], V(r)_{\chi,\mathfrak{p}}) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V(r-1)^{+}_{\chi,\mathfrak{p}} \xrightarrow{\sim} E_\chi$$

Definition 3.3. — The *p-adic fundamental line* $\Delta_{S,\mathfrak{p}}(f\otimes\chi)(r)$ is the one-dimensional E_χ-vector space

$$\Delta_{S,\mathfrak{p}}(f\otimes\chi)(r) \overset{\mathrm{def}}{=} \mathrm{Det}^{-1}_{E_\chi} \mathrm{R}\Gamma_{\mathrm{et}}\,(\mathbb{Z}[1/S], V(r)_{\chi,\mathfrak{p}}) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V(r-1)^{+}_{\chi,\mathfrak{p}}$$

together with the free of rank one sub-$\mathcal{O}_\chi$-module

$$\Delta_{S,\mathfrak{p}}(T(f)\otimes\chi)(r) \overset{\mathrm{def}}{=} \mathrm{Det}^{-1}_{\mathcal{O}_\chi} \mathrm{R}\Gamma_{\mathrm{et}}\,(\mathbb{Z}[1/S], T(r)_{\chi,\mathfrak{p}}) \otimes_{\mathcal{O}_\chi} \mathrm{Det}^{-1}_{\mathcal{O}_\chi} T(r-1)^{+}_{\chi,\mathfrak{p}}$$

obtained by choosing a $G_{\mathbb{Q}}$-stable $\mathcal{O}_\chi$-lattice T inside $V_{\chi,\mathfrak{p}}$.

It is well-known that $\Delta_{S,\mathfrak{p}}(T(f)\otimes\chi)(r)$ does not depend on the choice of T (see [18, 4]).

***Conjecture 3.4* (The Tamagawa Number Conjecture, [1, 18]).** — *The S-partial zeta morphism $z_S(f\otimes\chi)(r)$ of $M(f\otimes\chi)(r)$ exists and*

$$z_S(f\otimes\chi)(r) : \Delta_{S,\mathfrak{p}}(f\otimes\chi)(r) \xrightarrow{\sim} E_\chi$$

induces an isomorphism

$$z_S(f\otimes\chi)(r) : \Delta_{S,\mathfrak{p}}(T(f)\otimes\chi)(r) \xrightarrow{\sim} \mathcal{O}_\chi.$$

Conjecture 3.4 is equivalent to the Tamagawa Number Conjecture for the motive $M(f\otimes\chi)(r)$ of [1] expressing the value at zero of the L-function of $M(f\otimes\chi)(r)$ in terms of Tamagawa measures (see [12, Section 11.6]).

3.3. The Iwasawa Main Conjecture. — Fix $T(f)$ a $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice inside $M(f)_{\mathrm{et},\mathfrak{p}}$. Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, that is to say the union of the finite extensions $\mathbb{Q}_n$ for all $n\in\mathbb{N}$. Let Λ be the power series-ring $\mathcal{O}[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$. The ring Λ seen as module over itself is endowed with a Galois action by the composition of the quotient $G_{\mathbb{Q}}\twoheadrightarrow\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})\hookrightarrow\Lambda^\times$. Write $T(f)_{\mathrm{Iw}}$ for $T(f)\otimes_{\mathcal{O}}\Lambda$ with $G_{\mathbb{Q}}$-action on both sides of the tensor product. Note that $T(f)_{\mathrm{Iw}}$ is a Galois representation with coefficients in Λ which is unramified outside of Σ and such that $T(f)_{\mathrm{Iw}}\otimes_\Lambda k$ is equal to ρ_f. Hence, $T(f)_{\mathrm{Iw}}$ is a deformation of $\bar\rho_f$ and hence a Λ-valued point of $\mathscr{X}_\Sigma(\bar\rho)$. More generally, if $\mathcal{P}\in\operatorname{Spec}\Lambda$ is a prime ideal of height 1 prime to (p), then $\Lambda/\mathcal{P}$ is a discrete valuation ring flat over $\mathbb{Z}_p$ and $T(f)_{\mathrm{Iw}}\otimes_\Lambda\Lambda/\mathcal{P}$ is a Galois representation deforming $\bar\rho_f$ and hence an $\Lambda/\mathcal{P}$-valued point of $\mathscr{X}_\Sigma(\bar\rho)$. Hence, $\operatorname{Spec}\Lambda[1/p]$ is a subset of $\mathscr{X}_\Sigma(\bar\rho)$. As we already observed at the end of the second paragraph of the proof of Theorem 2.1, Λ is the universal deformation of the trivial character with values in $k^\times$ and so the subset $\operatorname{Spec}\Lambda$ is closed in $\mathscr{X}_\Sigma(\bar\rho)$.
Consider a morphism

$$Z_{\mathrm{Iw}}:T(f)_{\mathrm{Iw}}(-1)^+\longrightarrow H^1_{\mathrm{et}}\left(\mathbb{Z}[1/S],T(f)_{\mathrm{Iw}}\right). \tag{6}$$

After descent to a finite level n, a choice of character χ, a choice of a twist r such that $M(f\otimes\chi)(r)$ is strictly critical and inversion of p, the morphism (6) induces a morphism

$$Z_{\chi,r}:V(r-1)^+_{\chi,\mathfrak{p}}\overset{\sim}{\longrightarrow}H^1_{\mathrm{et}}(\mathbb{Z}[1/S],V(r)_{\chi,\mathfrak{p}}).$$

Definition 3.5. — A morphism

$$Z_{\mathrm{Iw}}:T(f)_{\mathrm{Iw}}(-1)^+\longrightarrow H^1_{\mathrm{et}}\left(\mathbb{Z}[1/S],T(f)_{\mathrm{Iw}}\right). \tag{7}$$

is the *S-partial zeta morphism* of $T(f)_{\mathrm{Iw}}$ if the morphism $Z_{\chi,r}$ it induces coincides with the S-partial zeta morphism $\mathbf{z}_S(f\otimes\chi)(r)$ for all choice of χ and r such that $M(f\otimes\chi)(r)$ is strictly critical.

The following deep theorem of K.Kato establishes that $T(f)_{\mathrm{Iw}}$ has an S-partial zeta morphism.

***Theorem 3.6* (K.Kato).** — *For all $S\supset\{p\}$, there exists an S-partial zeta morphism*

$$\mathbf{z}(f)_{\mathrm{Iw}}:T(f)_{\mathrm{Iw}}(-1)^+\longrightarrow H^1_{\mathrm{et}}\left(\mathbb{Z}[1/S],T(f)_{\mathrm{Iw}}\right)$$

and by descent an S-partial zeta morphism

$$\mathbf{z}_S(f\otimes\chi)(r):V(r-1)^+_{\chi,\mathfrak{p}}\overset{\sim}{\longrightarrow}H^1_{\mathrm{et}}(\mathbb{Z}[1/S],V(r)_{\chi,\mathfrak{p}})$$

provided $M(f\otimes\chi)(r)$ is strictly critical.

Proof. — See [21, Theorem 12.4,12.5]. □

Put

$$\Delta(T(f)_{\mathrm{Iw}})\overset{\mathrm{def}}{=}\operatorname{Det}^{-1}_\Lambda\mathrm{R}\Gamma_{\mathrm{et}}\left(\mathbb{Z}[1/S],T(f)_{\mathrm{Iw}}\right)\otimes_\Lambda\operatorname{Det}^{-1}_\Lambda T(f)_{\mathrm{Iw}}(-1)^+.$$

Then $\Delta(T(f)_{\mathrm{Iw}})$ is a free Λ-module of rank 1. Note that $\Delta(T(f)_{\mathrm{Iw}})$ is indeed well-defined as $\mathrm{R}\Gamma_{\mathrm{et}}\left(\mathbb{Z}[1/S],T(f)_{\mathrm{Iw}}\right)$ and $T(f)_{\mathrm{Iw}}(-1)^+$ are bounded complexes of Λ-modules, hence perfect complexes of Λ-modules by the theorem of Auslander–Buchsbaum and Serre as Λ is a regular local ring. Exactly as in the case of $\mathcal{O}_\chi$, the morphism (6) induces an isomorphism

$$Z_{\mathrm{Iw}}:\Delta(T(f)_{\mathrm{Iw}})\otimes_\Lambda\mathrm{Frac}(\Lambda)\overset{\sim}{\longrightarrow}\mathrm{Frac}(\Lambda)$$

The following is the statement of the Iwasawa Main Conjecture for the motive $M(f)$.

***Conjecture 3.7* (The Iwasawa Main Conjecture).** — *The S-partial zeta morphism $\mathbf{z}(f)_{\mathrm{Iw}}$ of $T(f)_{\mathrm{Iw}}$ induces an isomorphism*

$$\mathbf{z}(f)_{\mathrm{Iw}} : \Delta(T(f)_{\mathrm{Iw}}) \xrightarrow{\sim} \Lambda.$$

When the eigencuspform f is nearly ordinary at p, Conjecture 3.7 implies the usual formulation of the Iwasawa Main Conjecture in terms of p-adic L-function and Selmer modules.

Proposition 3.8. — *Assume that f is nearly ordinary at p and that Conjecture 3.7 holds. Then there is an equality*

$$\left(L_p^{\mathrm{cyc}}(f)\right)\Lambda = \left(\mathrm{char}_\Lambda\, \widetilde{H}^2_f(G_{\mathbb{Q},\Sigma}, T(f)_{\mathrm{Iw}})\right) \tag{8}$$

between the ideal generated by the Mazur p-adic L-function $L_p^{\mathrm{cyc}}(f) \in \Lambda$ and the characteristic ideal of the second cohomology group of the Nekovář–Selmer complex $\mathrm{R}\Gamma_f(G_{\mathbb{Q},\Sigma}, T(f)_{\mathrm{Iw}})$ with Greenberg's local condition at p (see [24] *for the definition of L_p^{cyc} and* [26] *for the definition of $\mathrm{R}\Gamma_f(G_{\mathbb{Q},\Sigma}, T(f)_{\mathrm{Iw}})$).*

Note that this holds even if $L_p^{\mathrm{cyc}}(f)$ has an exceptional zero.

Proof. — When in addition to being nearly ordinary, the $\mathrm{Gl}_2(\mathbb{Q}_p)$-representation $\pi(f)_p$ is in the principal series (this is for instance the case if $k > 2$), this follows from the arguments of [21, Section 17.13]. The remaining case, that is to say if $\pi(f)_p$ is Steinberg, is treated in [5, Section 4.4]. □

When f is of weight 2 or when $a_p(f) = 0$, there are conjectures of R. Pollack, S. Kobayashi and I. Sprung predicting equalities between a pair of ideals generated in Λ by a pair of p-adic L-functions on one hand and the characteristic ideals of a pair of Λ_{Iw}-adic Selmer groups on the other hand. When they are known to make sense, these conjectures also follow from Conjecture 3.7 (see [22, 29, 35] for formulations and details).
The following proposition is easy but it does not appear in the standard literature on the topic and is the prototype of the results we prove in this text.

Proposition 3.9. — *The Iwasawa Main Conjecture (Conjecture 3.7) implies the Tamagawa Number Conjecture (Conjecture 3.4) for all integers $1 \leq r \leq k-1$ and all finite order character χ of p-power order.*

Proof. — This boils down to the commutativity of the following diagram

$$\begin{array}{ccc} \Delta(T(f)_{\mathrm{Iw}}) & \xrightarrow{\mathbf{z}(f)_{\mathrm{Iw}}} & \Lambda \\ \downarrow & & \downarrow \chi \\ \Delta_{S,\mathfrak{p}}(T(f)\otimes\chi)(r) & \xrightarrow{\mathbf{z}_S(f\otimes\chi)(r)} & \mathcal{O}_\chi \end{array}$$

where the left vertical arrows is the isomorphism

$$\Delta(T(f)_{\mathrm{Iw}}) \otimes_{\Lambda,\chi} \mathcal{O}_\chi \overset{\mathrm{can}}{\simeq} \Delta_{S,\mathfrak{p}}(T(f)\otimes\chi)(r)$$

induced by the canonical isomorphism of complexes

$$\mathrm{R}\Gamma_{\mathrm{et}}\left(\mathbb{Z}[1/S], T(f)_{\mathrm{Iw}}\right) \overset{\mathrm{L}}{\otimes}_{\Lambda,\chi} \mathcal{O}_\chi \simeq \mathrm{R}\Gamma_{\mathrm{et}}\left(\mathbb{Z}[1/S], T(f\otimes\chi)(r)\right)$$

coming from the universal coefficient property of étale cohomology (note that this uses crucially the fact that $T(f)_{\mathrm{Iw}}$ is a perfect complex of étale sheaves of Λ-modules on $\operatorname{Spec}\mathbb{Z}[1/S]$). □

3.4. The universal Iwasawa Main Conjecture. — Let k be the residual field of $\mathcal{O}$. Henceforth, we assume that the residual representation $\bar{\rho}: G_{\mathbb{Q}} \to \mathrm{Gl}_2(k)$ attached to f satisfies the two assumptions of Theorem 1.1 that we repeat here.

Assumption 3.10. —

1. *The $G_{\mathbb{Q}}$-representation $\bar{\rho}$ is isomorphic to $\bar{\rho}_f$ and its image contains a subgroup conjugated to* $\mathrm{SL}_2(\mathbb{F}_p)$.
2. *If $(\bar{\rho}|G_{\mathbb{Q}_p})^{ss} \simeq \bar{\chi}_1 \oplus \bar{\chi}_2$, then $\bar{\chi}_1^{-1}\bar{\chi}_2 \notin \{1, \bar{\chi}_{\mathrm{cyc}}^{\pm 1}\}$.*

We consider the universal deformation (T_Σ, ρ_Σ) of Section 2 and recall that $R_\Sigma(\bar{\rho})$ is a reduced, complete, local, noetherian, complete intersection ring of relative dimension 3.
The following theorem, which is due to K. Nakamura, is an essential ingredient in our proof.

Theorem 3.11 **([25], Theorem 1.1).** — *There exists a zeta morphism*

$$\mathbf{z}_\Sigma : T_\Sigma(-1)^+ \longrightarrow H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma)$$

such that for all classical point λ_f the following diagram commutes

$$\begin{array}{ccc} T_\Sigma(-1)^+ & \xrightarrow{\quad \mathbf{z}_\Sigma \quad} & H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma) \\ \downarrow & & \downarrow \\ T(f)_{\mathrm{Iw}}(-1)^+ & \xrightarrow{\quad \mathbf{z}(f)_{\mathrm{Iw}} \quad} & H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T(f)_{\mathrm{Iw}}) \end{array} \tag{9}$$

where $\mathbf{z}(f)_{\mathrm{Iw}}$ is the zeta morphism of Theorem 3.6.

We recall that this deep theorem requires the full strength of the p-adic Langlands Correspondence of [6, 28].

Definition 3.12. — The universal fundamental line is the rank-one, free $R_\Sigma(\bar{\rho})$-module Δ_Σ defined by

$$\Delta_\Sigma \stackrel{\text{def}}{=} \operatorname{Det}^{-1}_{R_\Sigma(\bar{\rho})} \mathrm{R}\Gamma_{\mathrm{et}}\left(\mathbb{Z}[1/\Sigma], T_\Sigma\right) \otimes_{R_\Sigma(\bar{\rho})} \operatorname{Det}^{-1}_{R_\Sigma(\bar{\rho})} T_\Sigma(-1)^+.$$

Here again, note that T_Σ is a perfect complex of étale sheaves of $R_\Sigma(\bar{\rho})$-module so that Δ_Σ is well-defined (and commutes with arbitrary change of coefficients).

Proposition 3.13. — *Let $Q(R_\Sigma(\bar{\rho}))$ be the total ring of quotients of $R_\Sigma(\bar{\rho})$. The morphism $\mathbf{z}_\Sigma$ induces*

$$\mathbf{z}_\Sigma : \Delta_\Sigma \otimes_{R_\Sigma(\bar{\rho})} Q(R_\Sigma(\bar{\rho})) \xrightarrow{\sim} Q(R_\Sigma(\bar{\rho})).$$

Proof. — As the determinant of the zero complex is canonically identified with the coefficient ring, it suffices to show that the $R_\Sigma(\bar{\rho})$-modules $H^2_{\mathrm{et}}\left(\mathbb{Z}[1/\Sigma], T_\Sigma\right)$ and $H^1_{\mathrm{et}}\left(\mathbb{Z}[1/\Sigma], T_\Sigma\right) / \operatorname{Im}\mathbf{z}_\Sigma$ are torsion modules. The canonical isomorphism

$$\mathrm{R}\Gamma_{\mathrm{et}}\left(\mathbb{Z}[1/\Sigma], T_\Sigma\right) \overset{\mathrm{L}}{\otimes}_{R_\Sigma(\bar{\rho}),x} \Lambda \stackrel{\text{can}}{\simeq} \mathrm{R}\Gamma_{\mathrm{et}}\left(\mathbb{Z}[1/\Sigma], T(f)_{\mathrm{Iw}}\right)$$

and Nakayama's lemma show first that the cohomology of $\mathrm{R}\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p], T_\Sigma)$ (which is *a priori* concentrated in degree $[0,3]$) is concentrated in degree $[1,2]$, then that $H^2_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma)$ is torsion and finally that $H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma)$ is generated by at most one element. In order to show that $H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma) / \operatorname{Im} \mathbf{z}_\Sigma$ is torsion, it is consequently sufficient to do so after base-changing to each irreducible component of $R_\Sigma(\bar\rho)$, in which case it is sufficient to show that $H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T(f)_{\mathrm{Iw}}) / \operatorname{Im} \mathbf{z}(f)_{\mathrm{Iw}}$ is torsion at a classical point (which we know exist on the irreducible component under examination by Theorem 2.1). The result then follows by [21, Theorem 12.4]. □

Proposition 3.13 shows that the following conjecture is at least meaningful.

***Conjecture 3.14* (Universal Iwasawa Main Conjecture).** — *The universal zeta morphism induces*

$$\mathbf{z}_\Sigma : \Delta_\Sigma \xrightarrow{\sim} R_\Sigma(\bar\rho).$$

Let λ be a point of $\mathscr{X}_\Sigma(\bar\rho)[1/p]$ with values in a reduced ring A and let us write T_λ for $T_\Sigma \otimes_{R_\Sigma(\bar\rho),\lambda} A$. We say that T_λ is Iwasawa-suitable if the complex

$$\operatorname{Cone}\Big(T_\lambda(-1)^+[-1] \longrightarrow \mathrm{R}\Gamma_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\lambda)\Big)$$

built out of the morphism

$$\mathbf{z}_\Sigma(\lambda) : T_\lambda(-1)^+ \longrightarrow H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\lambda)$$

induced from $\mathbf{z}_\Sigma$ is acyclic after extension of scalars to the total ring of quotient $Q(A)$ of A. Proposition 3.13 shows that the identity is Iwasawa-suitable and the main results of [21] imply in particular that $\lambda(f) : R_\Sigma(\bar\rho) \to \Lambda$ given by the system of eigenvalues of a classical eigencuspform is Iwasawa-suitable.
The following seemingly formal and easy proposition is the key to the proof of our theorem. As explained in the introduction, it can be interpreted as an optimal version of Greenberg's Control Theorem ([16, 27, 14, 13]). It also appears to be false in general and very hard to prove even with supplementary assumptions if fundamental lines are replaced by characteristic ideals of Selmer groups (in particular, our crucial reliance on this proposition justifies the use of fundamental lines rather than the more concrete and usual Selmer modules).

Proposition 3.15. — *Let*

$$\begin{array}{ccc} R_\Sigma(\bar\rho) & \longrightarrow & B \\ {\scriptstyle\lambda}\downarrow & \nearrow_{\psi} & \\ A & & \end{array}$$

be a commutative diagram of Iwasawa-suitable ring-morphisms (in practice, λ is often the identity or the projection to an irreducible component and ψ is a classical point or a point with good properties). We define

$$\Delta_\Sigma(T_\lambda) = \operatorname{Det}^{-1}_A \mathrm{R}\Gamma_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\lambda) \otimes_A \operatorname{Det}^{-1}_A T_\lambda(-1)^+$$

and

$$\Delta_\Sigma(T_\psi) = \operatorname{Det}^{-1}_B \mathrm{R}\Gamma_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\psi) \otimes_B \operatorname{Det}^{-1}_B T_\psi(-1)^+.$$

Let $(x,y)\in A^2$ be such that the image of the zeta morphism $\mathbf{z}_\Sigma(\lambda)$ is equal to $\frac{x}{y}A$ and let likewise $(x',y')\in B^2$ be such that the image of the zeta morphism $\mathbf{z}_\Sigma(\psi)$ is equal to $\frac{x'}{y'}B$. Then the canonical isomorphism

$$\Delta_\Sigma(T_\lambda)\otimes_{A,\psi}B\overset{\sim}{\longrightarrow}\Delta_\Sigma(T_\psi)$$

fits into a commutative diagram

$$\begin{array}{ccc}\Delta_\Sigma(T_\lambda) & \xrightarrow{\mathbf{z}_\Sigma(\lambda)} & \frac{x}{y}A \\ {\scriptstyle -\otimes_{A,\psi}B}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ \Delta_\Sigma(T_\psi) & \xrightarrow{\mathbf{z}_\Sigma(\psi)} & \frac{x'}{y'}B\end{array}\;.$$

In particular, the morphism ψ extends to a morphism $\frac{x}{y}A\to\frac{x'}{y'}B$.

Proof. — Let $\mathfrak{p}\subset\operatorname{Spec}A$ be the kernel of $\psi:A\to B$. The complex $\mathrm{R}\Gamma_{\mathrm{et}}(\mathbb{Z}[1/\Sigma],T_\lambda\otimes_A A_\mathfrak{p})$ is perfect and commutes with $-\overset{\mathrm{L}}{\otimes}_{A_\mathfrak{p}}\kappa(\mathfrak{p})$. As the complex

$$\operatorname{Cone}\left((T_\lambda\otimes_A\kappa(\mathfrak{p}))(-1)^+\overset{\mathbf{z}(\lambda)}{\longrightarrow}\mathrm{R}\Gamma_{\mathrm{et}}(\mathbb{Z}[1/\Sigma],T_\lambda\otimes_A\kappa(\mathfrak{p}))\right)$$

is acyclic by definition of λ being Iwasawa-suitable, the complex

$$\operatorname{Cone}\left((T_\lambda\otimes_A A_\mathfrak{p})(-1)^+\overset{\mathbf{z}(\lambda)}{\longrightarrow}\mathrm{R}\Gamma_{\mathrm{et}}(\mathbb{Z}[1/\Sigma],T_\lambda\otimes_A A_\mathfrak{p})\right)$$

is acyclic by Nakayama's lemma. By functoriality of Det, there is a commutative diagram

$$\begin{array}{ccccc}\Delta_\Sigma(T_\lambda) & \hookrightarrow & \Delta_A(T_\lambda\otimes_A A_\mathfrak{p}) & \overset{\sim}{\longrightarrow} & A_\mathfrak{p} \\ {\scriptstyle -\otimes_A B}\big\downarrow & & {\scriptstyle -\otimes_{A_\mathfrak{p}}\kappa(\mathfrak{p})}\big\downarrow & & \big\downarrow{\scriptstyle -\otimes_{A_\mathfrak{p}}\kappa(\mathfrak{p})} \\ \Delta_\Sigma(T_\psi) & \hookrightarrow & \Delta_\Sigma(T_\psi\otimes\kappa(\mathfrak{p})) & \overset{\sim}{\longrightarrow} & \kappa(\mathfrak{p})\end{array}$$

whose vertical arrows are induced by ψ. In particular, if $\mathbf{z}_\Sigma(\lambda)\,(\Delta_\Sigma(T_\lambda))$ is generated by x/y, then y may be chosen so that it does not belong to $\mathfrak{p}$ and $\psi(x)/\psi(y)$ then generates $\mathbf{z}_\Sigma(\psi)\,(\Delta_\Sigma(T_\psi))$. □

We deduce the following theorem.

Theorem 3.16. — *The Universal Iwasawa Main Conjecture (Conjecture 3.14) implies the Iwasawa Main Conjecture (Conjecture 3.7) at all classical points.*

Proof. — This is Proposition 3.15 applied to λ equal to the identity and ψ equal to the system of eigenvalues attached to a classical eigencuspform. □

We also record the following weaker version of Conjecture 3.14, which will prove useful in the following. As $R_\Sigma(\bar\rho)$ is a Cohen–Macaulay ring of relative dimension zero over $\mathbf{\Lambda}=\mathcal{O}[\![X_1,X_2,X_3]\!]$, it is a free $\mathbf{\Lambda}$-module of rank d and T_Σ is a $\mathbf{\Lambda}[G_{\mathbb{Q},\Sigma}]$-module which is free of rank $2d$ as $\mathbf{\Lambda}$-module. We define $\Delta_{\mathbf{\Lambda}}$ as

$$\Delta_{\mathbf{\Lambda}}\overset{\mathrm{def}}{=}\operatorname{Det}^{-1}_{\mathbf{\Lambda}}\mathrm{R}\Gamma_{\mathrm{et}}\left(\mathbb{Z}[1/\Sigma],T_\Sigma\right)\otimes_\Lambda\operatorname{Det}^{-1}_{\mathbf{\Lambda}}T_\Sigma(-1)^+.$$

Since $H^2_{\mathrm{et}}(\mathbb{Z}[1/\Sigma],T_\Sigma)$ is a torsion $\mathbf{\Lambda}$-module by Nakayama's lemma and since $T_\Sigma(-1)^+$ and $H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma],T_\Sigma)$ are torsion-free $\mathbf{\Lambda}$-modules of the same rank (namely d), we may then view $\mathbf{z}_\Sigma$ as a morphism

$$\mathbf{z}_\Sigma:\Delta_{\mathbf{\Lambda}}\otimes_{\mathbf{\Lambda}}\operatorname{Frac}(\mathbf{\Lambda})\overset{\sim}{\longrightarrow}\operatorname{Frac}(\mathbf{\Lambda}).$$

***Conjecture 3.17* (Universal Iwasawa Main Conjecture in families).** — *The universal zeta morphism induces an isomorphism*

$$\mathbf{z}_\Sigma:\Delta_{\mathbf{\Lambda}}\overset{\sim}{\longrightarrow}\mathbf{\Lambda}.$$

We note that contrary to other comparable results, the compatibility of Conjecture 3.17 with Conjecture 3.7 does not appear to be formal. Indeed, it is an integral part of the statement of our main theorem and we record the fact that we don't know how to prove this compatibility without assumption 1 of Theorem 1.1 (whereas this assumption plays no role in our other compatibility results).

4. Proof of the main theorem and examples

4.1. Proof of Theorem 1.1. — In this section, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. — We show that the Iwasawa Main Conjecture for all points in a fiber of $x\in\mathscr{X}^{\mathrm{sm}}_\Sigma(\bar\rho)$ entails Conjecture 3.17 (implication iv. $\Longrightarrow$ i. of the theorem) and that Conjecture 3.17 entails the Iwasawa Main Conjecture for all classical points (implication i. $\Longrightarrow$ ii.). The remaining implications (ii. $\Longrightarrow$ iii. and iii. $\Longrightarrow$ iv.) are formal consequences of the definitions.
We first show that $\mathbf{z}_\Sigma$ sends $\Delta_{\mathbf{\Lambda}}^{-1}$ inside $\mathbf{\Lambda}$. Assume by way of contradiction that this is not the case. Let $\lambda:R_\Sigma(\bar\rho)\to\Lambda$ be an Iwasawa-suitable specialization. According to Proposition 3.15, there is then a commutative diagram

$$\begin{array}{ccc}\Delta_{\mathbf{\Lambda}}^{-1} & \xrightarrow{\ \mathbf{z}_\Sigma\ } & \frac{x}{y}\mathbf{\Lambda}\\ \downarrow{\scriptstyle -\otimes_{\mathbf{\Lambda}}\Lambda} & & \downarrow{\scriptstyle\lambda}\\ \Delta_\Sigma(T_\lambda)^{-1} & \xrightarrow{\ \mathbf{z}_\Sigma(\lambda)\ } & \frac{x'}{y'}\Lambda.\end{array}$$

Our hypothesis is that x/y does not belong to $\mathbf{\Lambda}$. As $\mathbf{\Lambda}$ is a factorial ring, there then exists $\mathfrak{p}\in\operatorname{Spec}\mathbf{\Lambda}$ a height-one prime containing y but not x. As $\mathbf{\Lambda}$ is regular, $\mathfrak{p}$ is principal. Let y_0 be a choice of one of its generator. We may choose λ such that $\lambda(y_0)$ belongs to $\mathfrak{m}_\Lambda^n$ but $\lambda(x)$ does not belong to $\mathfrak{m}_\Lambda^n$. Removing a Zariski-closed subset of large codimension, we may further assume that $\operatorname{Spec}R_\Sigma(\bar\rho)$ is étale over $\operatorname{Spec}\mathbf{\Lambda}$ at λ and such that all specializations in the fiber above the point of $\mathbf{\Lambda}$ below λ are Iwasawa-suitable. Because $\operatorname{Spec}\mathbf{T}^\Sigma_{\mathfrak{m}_{\bar\rho}}$ is étale over $\operatorname{Spec}\mathbf{\Lambda}$, enlarging $\mathcal{O}$ if necessary, we may assume that T_λ is a lattice inside a direct sum

$$\bigoplus_{r=1}^{d}V_{\lambda,r}$$

of $G_{\mathbb{Q},\Sigma}$-representations $V_{\lambda,r}$ which are free modules of rank 2 over $\mathrm{Frac}(\Lambda)$. There is thus a short exact sequence

$$0 \longrightarrow T_\lambda \longrightarrow \bigoplus_{r=1}^{d} T_{\lambda,r} \longrightarrow C \longrightarrow 0 \tag{10}$$

where $T_{\lambda,r}$ are specializations of T_Σ attached to specializations $\lambda_r : R_\Sigma(\bar\rho) \to \Lambda$ and where C is a torsion Λ-module. The short exact sequence (10) induces a canonical isomorphism

$$\bigotimes_{r=1}^{d} \Delta_\Sigma(T_{\lambda,r}) \overset{\mathrm{can}}{\simeq} \Delta_\Sigma(T_\lambda) \otimes_\Lambda \Delta_\Sigma(C). \tag{11}$$

According to [4, Proposition 1.20] (taking into account the duality between $\mathrm{R}\Gamma_c$ and $\mathrm{R}\Gamma_{\mathrm{et}}$, see for instance [37, Appendix]), $\Delta_\Sigma(C)$ is the unit object in the category of graded invertible modules. Hence

$$\mathbf{z}_\Sigma(\lambda)\left(\bigotimes_{r=1}^{d} \Delta_\Sigma(T_{\lambda,r})^{-1}\right) \not\subset \Lambda.$$

By construction,

$$\mathbf{z}_\Sigma(\lambda) = \bigotimes_{r=1}^{d} \mathbf{z}_\Sigma(\lambda_r)$$

so there must exists at least one r_0 such that

$$\mathbf{z}_\Sigma(\lambda_{r_0})\left(\Delta_\Sigma(T_{\lambda,r_0})^{-1}\right) \not\subset \Lambda.$$

By our choice of λ, λ_{r_0} is Iwasawa-suitable. This is a contradiction with [20, Theorem 0.8 and Proposition 6.7]. Hence $\mathbf{z}_\Sigma$ sends $\Delta_{\boldsymbol{\Lambda}}^{-1}$ inside $\boldsymbol{\Lambda}$.
According to our assumption, there is a fiber $S_x = \{x_i | 1 \le i \le d\}$ of a point $x \in \mathscr{X}_\Sigma^{\mathrm{sm}}(\bar\rho)$ such that the Iwasawa Main Conjecture is true at all the classical points x_i. According to Proposition 3.15 and to what we just proved, there then exists $\alpha \in \boldsymbol{\Lambda}$ and a commutative diagram

$$\begin{array}{ccc}
\Delta_{\boldsymbol{\Lambda}}^{-1} & \xrightarrow{\quad \mathbf{z}_\Sigma \quad} & \alpha\boldsymbol{\Lambda} \\
\Big\downarrow{\scriptstyle -\otimes_{\boldsymbol{\Lambda},x}\Lambda} & & \Big\downarrow{\scriptstyle \lambda} \\
\bigotimes_{\lambda_i \in S_x} \Delta_\Sigma(T_{\lambda_i})^{-1} & \xrightarrow{\bigotimes_{\lambda_i \in S_x} \mathbf{z}_\Sigma(\lambda_i)} & \Lambda
\end{array}$$

This implies that α is a unit and hence that $\mathbf{z}_\Sigma$ induces an isomorphism $\Delta_{\boldsymbol{\Lambda}} \overset{\sim}{\longrightarrow} \boldsymbol{\Lambda}$. Hence Conjecture 3.17 holds.

Now let $\lambda(f)$ be a classical point of $R_\Sigma(\bar\rho)$. According to [21, Theorem 12.5] and Proposition 3.15, there is a commutative diagram

$$\begin{array}{ccc} \Delta_\Sigma & \xrightarrow{\ \mathbf{z}_\Sigma\ } & \frac{\alpha}{\beta}R_\Sigma(\bar\rho) \\ {\scriptstyle -\otimes_{R_\Sigma(\bar\rho),\lambda(f)}\Lambda}\downarrow & & \downarrow{\scriptstyle \lambda(f)} \\ \Delta_\Sigma(T(f)_{\mathrm{Iw}}) & \xrightarrow{\ \mathbf{z}_\Sigma(f)_{\mathrm{Iw}}\ } & \frac{1}{\gamma}\Lambda \end{array}$$

in which we may choose β such that $\lambda(f)(\beta)\neq 0$. If on the other hand ψ belongs to $\mathscr{X}^{\mathrm{sm}}$, then the diagram

$$\begin{array}{ccc} \Delta_{\mathbf{\Lambda}} & \xrightarrow[\sim]{\ \mathbf{z}_\Sigma\ } & \mathbf{\Lambda} \\ {\scriptstyle -\otimes_{\mathbf{\Lambda},x}\Lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ \bigotimes_{\psi_i\in S_x}\Delta_\Sigma(T_{\psi_i})^{-1} & \xrightarrow{\ \bigotimes_{\lambda_i\in S_x}\mathbf{z}_\Sigma(\psi_i)\ } & \Lambda \end{array}$$

is commutative (in this diagram, the fact that the top horizontal arrow is an isomorphism is the statement of Conjecture 3.17, which we have seen holds under our ongoing hypotheses). Since $\mathbf{z}(\psi_i)$ sends $\Delta_\Sigma(T_{\psi_i})^{-1}$ inside Λ, this can happen only if we have a commutative diagram

$$\begin{array}{ccc} \Delta_\Sigma & \xrightarrow{\ \mathbf{z}_\Sigma\ } & \frac{\alpha}{\beta}R_\Sigma(\bar\rho) \\ {\scriptstyle -\otimes_{R_\Sigma(\bar\rho),\psi}\Lambda}\downarrow & & \downarrow{\scriptstyle \psi} \\ \Delta_\Sigma(T_\psi) & \xrightarrow{\ \mathbf{z}_\Sigma(\psi)\ } & \Lambda. \end{array}$$

for ψ itself. As we may choose $\psi\in\mathscr{X}^{\mathrm{sm}}_\Sigma(\bar\rho)$ arbitrarily p-adically close to $\lambda(f)$, this entails that $\psi(1/\gamma)$ is a unit. Hence γ is a unit. This means exactly that Conjecture 3.7 is true for f. □

4.2. Examples. — As we noted, it follows from the construction of $\mathscr{X}^{\mathrm{sm}}_\Sigma(\bar\rho)$ that this set contains all nearly-ordinary points if $\bar\rho|G_{\mathbb{Q}_p}$ is reducible and that all points in the fiber of a nearly-ordinary point of $R_\Sigma(\bar\rho)$ are nearly-ordinary. Let us then assume that there exists a nearly-ordinary point $x\in\mathscr{X}_\Sigma(\bar\rho)$ attached to an eigencuspform f for which it is known that the Iwasawa Main Conjecture holds and such that the algebraic and analytic μ-invariants of f vanish (see [11] for the definition of these μ-invariants). It then follows from the main results of [11] that the Iwasawa Main Conjecture holds for all nearly-ordinary points in $\mathscr{X}_\Sigma(\bar\rho)$, in particular for all points in the fiber of x. According to Theorem 1.1, the Iwasawa Main Conjecture is then true for all classical points in $\mathscr{X}_\Sigma(\bar\rho)$. As the set of nearly-ordinary points in $\mathscr{X}_\Sigma(\bar\rho)$ has positive codimension (it is of relative dimension 2 over $\mathcal{O}$), this means that the

results of [11] were not applicable to almost all points in $\mathscr{X}_\Sigma(\bar\rho)$, whereas Theorem 1.1 applies to all. In particular, one can take again the numerical examples of [11] and extend them to non-ordinary congruences.
Let us consider the following examples. To the elliptic curve

$$E : y^2 = x^3 + x - 10$$

is attached an eigencuspform $f \in S_2(\Gamma_0(52))$ with rational q-expansion and which is ordinary for $p = 5$ (indeed $a_5(f) = 2$). The residual representation attached to $E[p]$ satisfies the hypotheses of Theorem 1.1. Among the Galois conjugacy class of eigencuspforms in $S_{10}(\Gamma_0(52))$, we consider the one defined over the number field K defined as the splitting field of

$$x^5 + 294x^4 - 47204x^3 - 49484952x^2 - 442506240x + 491692032000.$$

In the ring of integers $\mathcal{O}_K$ of K, there are four primes over 5. For exactly two choices $\mathfrak{p}_i|p$ ($i \in \{1, 2\}$), we find eigencuspforms f_1 and f_2 such that f_i is congruent to f modulo $\mathfrak{p}_i$. The eigencuspform f_1 is $\mathfrak{p}_1$-ordinary whereas the eigencuspform f_2 has finite slope but is non-ordinary at $\mathfrak{p}_2$. In [17], it was shown that f has trivial algebraic and analytic μ-invariant and satisfies the Iwasawa Main Conjecture. It follows from [11] that the same is true for f_1 and from Theorem 1.1 that the Iwasawa Main Conjecture holds for f_2. Among eigencuspforms congruent to f, there is also $g \in S_2(\Gamma_0(1300))$ attached to the elliptic curve

$$E' : y^2 = x^3 + 625x - 6250,$$

which has bad additive reduction at $p = 5$ (the automorphic representation of $\mathrm{Gl}_2(\mathbb{Q}_p)$ attached to g is supercuspidal; in particular, g is not p-ordinary nor does it have finite slope at p). Theorem 1.1 applies to E' and shows that the Iwasawa Main Conjecture si true for the motive of g at $p = 5$ (as E' has additive reduction, no published result establishes this at present).

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Olivier Fouquet, Laboratoire de mathématiques de Besançon, 16, route de Gray 25000 Besançon, France
E-mail : olivier.fouquet@univ-fcomte.fr