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ON THE KERNEL OF THE GYSIN HOMOMORPHISM ON CHOW GROUPS OF ZERO CYCLES

by

Rina Paucar and Claudia Schoemann

Abstract. — Given a smooth projective connected surface over $\mathbb{C}$ embedded into a projective space $\mathbb{P}^d$ and a smooth projective curve C embedded into the surface we study the kernel of the Gysin homomorphism between the Chow groups of 0-cycles of degree zero of the curve and the surface induced by the closed embedding. Following the approach of Bannerjee and Guletskii we prove that the kernel of the Gysin homomorphism is a countable union of translates of an abelian subvariety A inside the Jacobian J of the curve C. We also prove that there is a c-open subset U_0 contained in the set $U \subset (\mathbb{P}^d)^*$ parametrizing the smooth projective curves such that $A = 0$ or $A = B$ for all curves parametrized by U_0, where B is the abelian subvariety of J corresponding to the vanishing cohomology $H^1(C, \mathbb{Q})_{\mathrm{van}}$ of C.
We give a background of algebraic cycles, Chow groups, Hodge structures, the Abel–Jacobi map, Lefschetz pencils and the irreducibility of the monodromy representation.

Résumé. — Étant donné une surface lisse projective connexe sur $\mathbb{C}$ plongée dans un espace projectif $\mathbb{P}^d$ et C une courbe lisse projective plongée dans la surface, on étudie le noyau de l'homomorphisme de Gysin entre les groupes de Chow des 0-cycles de degré zéro de la courbe et de la surface induit par l'injection fermée. Suivant l'approche de Bannerjee and Guletskii, on démontre que le noyau de l'homomorphisme de Gysin est une union dénombrable de translatées d'une sous-variété abélienne A dans la Jacobienne J de la courbe C. On démontre également qu'il existe un sous-ensemble c-ouvert U_0 de l'ensemble $U \subset (\mathbb{P}^d)^*$ paramétrisant les courbes lisses projectives tel que $A = 0$ ou $A = B$ pour toute courbe paramétrisée par U_0, où B est la sous-variété abélienne de J correspondant à la cohomologie évanescente $H^1(C, \mathbb{Q})_{\mathrm{van}}$ de C.
On donne une introduction aux cycles algébriques, aux groupes de Chow, aux structures de Hodge, à l'application d'Abel–Jacobi, aux pinceaux de Lefschetz et à l'irréductibilité de la représentation de monodromie.

1. Introduction

Let k be an algebraically closed field of characteristic 0, let S be a smooth projective surface over k, let $\mathsf{CH}_0(S)_{\deg=0}$ be the Chow group 0-cycles of degree zero, let alb_S be the Albanese morphism defined from $\mathsf{CH}_0(S)_{\deg=0}$ to the Albanese variety $\mathrm{Alb}(S)$ of S. Bloch's conjecture states that if S has geometric genus zero, then alb_S is an isomorphism, see [2] and [4].

Key words and phrases. — 0-cycles; rational, algebraic and homological equivalence; Chow groups; Hodge theory; Lefschetz pencils; the monodromy argument, Gysin homomorphism.

If the Kodaira dimension of S is < 2, i.e., S is of special type, Bloch's conjecture has been proven in [5]. If S has Kodaira dimension 2, i.e., S is of general type, the vanishing of the geometric genus of S implies the vanishing of the irregularity of S, and the conjecture simply states that any two points on S are rationally equivalent to each other. This is the hard case of Bloch's conjecture and only known for some particular cases.

In [1] Banerjee and Guletskii show a general version on the countability results of the Gysin kernel related to the countability results of the Gysin kernel for surfaces stated in [23, p. 304-305, Exercise 1 a,b]. They provide a formal and abstract proof based on the étale monodromy argument. Let us comment their approach. Let X be a smooth projective connected variety of dimension $2p$ embedded into $\mathbb{P}^m$ over an uncountable algebraically closed field k of characteristic 0, let Y be a hyperplane section of X, and let $\mathsf{A}^p(Y) = \frac{\mathsf{Z}^p(Y)_{\mathsf{alg}}}{\mathsf{Z}^p(Y)_{\mathsf{rat}}}$ (resp. $\mathsf{A}^{p+1}(X) = \frac{\mathsf{Z}^{p+1}(X)_{\mathsf{alg}}}{\mathsf{Z}^{p+1}(X)_{\mathsf{rat}}}$) be the continuous part of the Chow group $\mathsf{CH}^p(Y)$ (resp. $\mathsf{CH}^{p+1}(X)$), that is, algebraically trivial algebraic cycles modulo rational equivalence on Y (resp. on X). Whenever Y is smooth and satisfying three assumptions (the group $\mathsf{A}^p(Y)$ is regularly parametrized by an abelian variety A, $\mathsf{A}^p(Y) = \mathsf{CH}^p(Y)_{\deg=0}$ and $H^1_{et}(A, \mathbb{Q}_l(1-p)) \cong H^{2p-1}_{et}(Y, \mathbb{Q}_l)$, see §2 in [1]) they prove that the kernel of the Gysin pushforward homomorphism from $\mathsf{A}^p(Y)$ to $\mathsf{A}^{p+1}(X)$ induced by the closed embedding of Y into X is the union of a countable collection of shifts of a certain abelian subvariety A_0 inside A, and for a very general section Y either $A_0 = 0$ or A_0 coincides with an abelian subvariety A_1 in A. Due to their assumptions the case $p = 1$ of this result gives an approach to prove Bloch conjecture.

In this paper we study the Gysin kernel for the case of surfaces and we prove some results on the countability of the Gysin kernel related to the countability results of the Gysin kernel stated in [23, p. 304-305, Exercise 1a,b] and [1] which play an important role in the study of 0-cycles on surfaces, especially in the context of Bloch's conjecture. More precisely, let S be a connected smooth projective surface over $\mathbb{C}$. Let Σ be the complete linear system of a very ample divisor D on S and let $d = \dim(\Sigma)$. For any closed point $t \in \Sigma \cong \mathbb{P}^{d*}$, let H_t be the corresponding hyperplane in $\mathbb{P}^d$, $C_t = H_t \cap S$ the corresponding hyperplane section of S, and r_t the closed embedding of C_t into S. Let $U = \Sigma \setminus \Delta_S$ be the open subset of Σ parametrizing smooth hyperplane sections of S. Using properties of the Chow groups of 0-cycles of degree zero of the smooth hyperplane sections C_t of a surface S we prove that whenever C_t is a smooth hyperplane section, the Gysin kernel G_t, i.e., the kernel of the Gysin homomorphism r_{t*} from $\mathsf{CH}_0(C_t)_{\deg=0}$ to $\mathsf{CH}_0(S)_{\deg=0}$ induced by r_t, is the union of a countable collection of translates of an abelian subvariety A_t inside $B_t \subset J_t$, where A_t is the unique irreducible component passing through zero of the irredundant decomposition of G_t, and B_t is the abelian subvariety of the Jacobian J_t of the curve C_t corresponding to the Hodge substructure on $H^1(C_t, \mathbb{Z})_{\mathsf{van}}$. This result is the case $p = 1$ of [1, Theorem A] and it is also item a) of [23, p. 304, Exercise 1]. Then using the approach to prove Theorem B in [1], we present the proof that there is a c-open U_0 in U such that $A_t = 0$ for all $t \in U_0$ or $A_t = B_t$ for all $t \in U_0$, that is, we prove that for all $t \in U_0$ we have that A_t has only two possibilities and the same behaviour. We achieve this result because the fact that the curves C_t live in the c-open subset U_0 assures that they have a uniform behavior (i.e. that the fibers C_t with $t \in U_0$ are isomorphic to the geometric generic fiber of the family $f : \mathscr{C} \to \mathbb{P}^{d*}$ of hyperplane sections of S parametrized by $\Sigma = \mathbb{P}^{d*}$) and hence the subvarieties A_t with t in U_0 have also a uniform behaviour. This result is the case $p = 1$ of [1, Theorem B] and in order to explain the relation of this result with item b) of [23, p. 304, Exercise 1] we should say that Voisin in her book [23] uses the word *general* for properties holding in a c-open instead of using the word *very general* (see for example [23, §10]) which is more commonly used by convention. It is important to note that in order to prove the countability results on the Gysin kernel for the case of smooth projective connected variety of dimension $2p$ in [1] Banerjee and Guletskii make three assumptions called Assumption 1, Assumption 2, and Assumption 3. In this paper we also prove that these assumptions are not necessary for the case of surfaces because they turn out to be true facts which we prove and call Fact 1, Fact 2 and Fact 3 respectively. The plan of the paper is as follows. In Section 2 we give theoretical background about algebraic cycles, Chow groups, the notions of rational, algebraic and homological equivalence and their relations. In Section 3 we give the needed background about Hodge structures and the Abel–Jacobi map. In Section 4 we study Lefschetz pencils and the Monodromy representation. In Section 5 we state and prove the main result of the paper.

Notation

Unless otherwise stated, for a *scheme* we will mean an algebraic scheme over a field k (i.e. a scheme of finite type over k) which is separated; a *variety* will be an integral scheme; a *subvariety* of a scheme will be a closed subscheme that is a variety, a *point* on a scheme will be a closed point.

2. Algebraic Cycles

The purpose of this section is to recall some needed facts about intersection theory. We start with the definition of algebraic cycles, then we define rational equivalence which allows us to study some properties of the Chow groups $\mathsf{CH}_r(X)$ of r-cycles on a scheme X and in particular of $\mathsf{CH}_0(X)_{\deg=0}$, the Chow group of 0-cycles of degree zero, which is the main mathematical object of this paper, next we study the notion of algebraic equivalence which allows us to define the continuous part $\mathsf{A}_r(X)$ of the Chow group, i.e., the group of r-cycles algebraically equivalent to zero modulo the group of r-cycles rationally equivalent to zero. After that we study the notion of homological equivalence which allows us to define the group $\mathsf{CH}_r(X)_{\mathrm{hom}}$ of r-cycles homologically equivalent to zero modulo the group of r-cycles rationally equivalent to zero. We then study the relation between rational, algebraic and homological equivalence which allows to conclude that when X is a connected smooth projective variety over an algebraically closed field of characteristic zero $\mathsf{CH}_0(X)_{\deg=0}$, $\mathsf{A}_0(X)$ and $\mathsf{CH}_0(X)_{\mathrm{hom}}$ are isomorphic to each other which implies Fact 1 (see Lemma 2.49), therefore we gain a lot of information to study $\mathsf{CH}_0(X)_{\deg=0}$ via these isomorphisms. The main reference for this section is [7], alternatively see [18, 3, 6].

Definition 2.1 **(r-Cycle).** — Let X be a scheme over k. An *algebraic cycle of dimension* r on X or simply an *r-cycle* on X is a formal finite linear combination $Z = \sum n_i Z_i$, where $n_i \in \mathbb{Z}$, and Z_i are subvarieties of dimension r on X.

The set $\mathsf{Z}_r(X)$ of r-cycles on X is a free abelian group called the *group of r-cycles on X*.

Definition 2.2 **(Purely dimensional Scheme).** — Let $X_1, \ldots, X_t$ be the irreducible components of the scheme X. We say that X is *purely n-dimensional* if $\dim(X_i) = n$, for all i.

Remarks. —

1. If X is a purely n-dimensional scheme we have: $\mathsf{Z}_r(X) = \mathsf{Z}^{n-r}(X)$, where $\mathsf{Z}^{n-r}(X)$ is the *group of algebraic cycles of codimension* $n-r$ on X (see [18, §1.1.]).

2. If we want to work with linear combinations in a field F, we write $\mathsf{Z}_r(X)_F = \mathsf{Z}_r(X) \otimes_{\mathbb{Z}} F$ (see [18, §1.1.]).

Example 2.3. — Let X be a variety of dimension n.

1. 0-*cycles* on X are finite formal linear combination $Z = \sum n_i P_i$, where $n_i \in \mathbb{Z}$ and P_i are points of X.

2. *Cycles of codimension* 1 or $(n-1)$-*Cycles* or *divisors* on X are finite formal linear combination $Z = \sum n_i Z_i$, where $n_i \in \mathbb{Z}$ and Z_i are subvarieties of codimension 1 of X.

2.1. Rational equivalence. — Let X be a scheme.

Definition 2.4 **(r-cycle associated to a rational function).** — Let W be any subvariety of X of dimension $r+1$, and let $f \in k(W)^*$ be a nonzero rational function on W, then we can define a *r-cycle associated to f* on X as $\operatorname{div}(f) = \sum_V \mathsf{ord}_V(f) V$, where V runs over all subvarieties of codimension 1 on W, and $\mathsf{ord}_V(f)$ is the order of vanishing of f along V, see [7, §1.2].

The above definition holds for every subvariety W of codimension $r+1$ of X, see [7, §1.3] or [18, §1.2.].

Definition 2.5 **(r-cycle rationally equivalent to 0).** — A r-cycle Z on X is *rationally equivalent to zero*, denoted by $Z \sim_{\mathsf{rat}} 0$, if there is a finite number of $(r+1)$-dimensional subvarieties W_i of X and $f_i \in k(W_i)^*$ such that $Z = \sum \operatorname{div}(f_i)$.

The set $\mathsf{Z}_r(X)_{\mathsf{rat}} = \{Z \in \mathsf{Z}_r(X) : Z \sim_{\mathsf{rat}} 0\}$ of r-cycles rationally equivalent to 0 is a subgroup of $\mathsf{Z}_r(X)$.

Definition 2.6 **(Rationally equivalent cycles).** — Two r-cycles Z_1 and Z_2 on X are *rationally equivalent*, denoted by $Z_1 \sim_{\mathsf{rat}} Z_2$, if its difference $Z_1 - Z_2$ is rationally equivalent to 0.

Informally we say that two r-cycles Z and Z' on X are rationally equivalent if there exists a family of r-cycles on X parametrized by $\mathbb{P}^1$ interpolating between them. More precisely, if we restrict to smooth projective varieties we have the following alternative definition of rational equivalence, see for example [18, Lemma 1.2.5], [6, §1.2.2], [3, Introduction], and [10, Introduction].

Definition 2.7 **(Rationally equivalent cycles).** — Let X be a smooth projective variety. Two r-cycles Z and Z' on X are *rationally equivalent* if there exits $W \in \mathsf{Z}_{r+1}(X \times \mathbb{P}^1)$ such that for any $t \in \mathbb{P}^1$ defining by

$$W(t) := (\mathsf{pr}_X)_*(W \cdot (X \times \{t\}))$$

we have: $Z = W(t_1)$ and $Z' = W(t_2)$ for some $t_1, t_2 \in \mathbb{P}^1$. Here $\cdot$ is the intersection product, pr_X is the projection to X and $(\mathsf{pr}_X)_*$ is the pushforward homomorphism induced by pr_X, see [7, §1.4].

In terms of codimension the definition is as follows

***Definition 2.8* (Rationally equivalent cycles).** — Let X be a smooth projective variety. Two cycles Z and Z' of codimension i on X are *rationally equivalent* if there exits $W \in \mathsf{Z}^i(X \times \mathbb{P}^1)$ such that for any $t \in \mathbb{P}^1$ defining by $W(t) := (\mathsf{pr}_X)_*(W \cdot (X \times \{t\}))$, we have: $Z = W(t_1)$ and $Z' = W(t_2)$ for some $t_1, t_2 \in \mathbb{P}^1$.

2.2. Chow Groups. — Let X be a scheme.

***Definition 2.9* (Chow groups).** — The group quotient $\mathsf{CH}_r(X) = \frac{\mathsf{Z}_r(X)}{\mathsf{Z}_r(X)_{\mathsf{rat}}}$ of rational equivalence classes of r-cycles is called the *Chow group of r-cycles*.

***Theorem 2.10* (Rational equivalence pushes forward).** — *If $f : X \to Y$ is a proper morphism and Z is a r-cycle on X rationally equivalent to zero, then $f_*(Z)$ is a r-cycle rationally equivalent to zero on Y.*

Proof. — See [7, Theorem 1.4]. □

By Theorem 2.10 given a proper morphism $f : X \to Y$ there is an induced homomorphism on Chow groups $f_* : \mathsf{CH}_r(X) \to \mathsf{CH}_r(Y)$.

Definition 2.11. — The morphism f_* is called the *pushforward homomorphism* or *Gysin homomorphism* on Chow groups induced by f.

So, we see that the Chow groups have 'homological-like' properties.

Lemma 2.12. — *Let $f : X \to Y$ be flat morphism of relative dimension n and Z an a r-cycle on Y which is rationally equivalent to zero. Then $f^*(Z)$ is rationally equivalent to zero in $\mathsf{Z}_{r+n}(X)$.*

Proof. — See [7, Theorem 1.7]. □

By Lemma 2.12 there is an induced homomorphism on Chow groups

$$f^* : \mathsf{CH}_r(Y) \longrightarrow \mathsf{CH}_{r+n}(X).$$

Definition 2.13. — The morphism f^* is called *pullback homomorphism* on Chow groups induced by f.

So, we see that the Chow groups have also 'cohomological-like' properties.

The group $\mathsf{CH}_0(X)_{\deg=0}$. — Let X be a complete scheme over a field k, that is, X is proper over $\mathsf{Spec}(k)$.

***Definition 2.14* (Degree of a 0-cycle).** — The *degree of* 0*-cycles* on X is a homomorphism $\deg : \mathsf{Z}_0(X) \to \mathbb{Z}$, defined by $Z = \sum n_i P_i \mapsto \deg(Z) = \sum n_i [k(P_i) : k]$, where $k(P_i)$ denotes the residue field of the point P_i.

Claim. — $\deg = f_*$, where $f : X \to \mathsf{Spec}(k)$ is the structural morphism and f_* is the pushforward homomorphism on cycles induced by f (see [7, §1.4]).

Proof. — Since X is complete, the structure morphism $f : X \to \mathsf{Spec}(k)$ is proper, so there is an induced pushforward homomorphism $f_* : \mathsf{Z}_0(X) \to \mathsf{Z}_0(\mathsf{Spec}(k))$ defined by $f_*(Z) = \sum n_i f_*(P_i) = \sum n_i \deg(P_i/f(P_i))f(P_i)$
Since $\dim(P_i) = \dim(f(P_i)) = \dim(\mathsf{Spec}(k))$, we have $\deg(P_i/f(P_i)) = [k(P_i) : k(\mathsf{Spec}(k))] = [k(P_i) : k]$, then $f_*(Z) = \sum n_i[k(P_i) : k]\,\mathsf{Spec}(k)$. Sending $\mathsf{Spec}(k) \to 1$, we get $\deg = f_*$. ☐

By Theorem 2.10 we have an induced homomorphism, denoted also by deg, on Chow groups: $\deg : \mathsf{CH}_0(X) \to \mathsf{CH}_0(\mathsf{Spec}(k))$. Since $Z_0(\mathsf{Spec}(k)) = \mathsf{CH}_0(\mathsf{Spec}(k)) = \mathbb{Z}$ we get $\deg : \mathsf{CH}_0(X) \to \mathbb{Z}$ the degree homomorphism on the Chow group of 0-cycles.

***Definition 2.15* (The group $\mathsf{CH}_0(X)_{\deg=0}$).** — The kernel of $\deg : \mathsf{CH}_0(X) \to \mathbb{Z}$ is denoted by

$$\mathsf{CH}_0(X)_{\deg=0} = \mathsf{Ker}(\deg : \mathsf{CH}_0(X) \longrightarrow \mathbb{Z}).$$

This is the *Chow group of* 0*-cycles of degree zero* on X.

In particular, it follows that rationally equivalent cycles have the same degree.

Countability lemmas. — Let k be an uncountable field. In this subsection a variety is a reduced scheme, not necessarily irreducible.

Lemma 2.16. — *Let X be an irreducible quasi-projective algebraic variety over k. Then X can not be written as a countable union of its Zariski closed subsets, each of which is not the whole X.*

Proof. — See [1, Lemma 10]. ☐

***Definition 2.17* (Irredundant countable union).** — A countable union $V = \cup_{n\in\mathbb{N}} V_n$ of algebraic varieties will be called *irredundant* if V_n is irreducible for each n and $V_m \not\subset V_n$ for $m \neq n$. If V is a irredundant decomposition, then the sets V_n are called c-components of V.

Lemma 2.18. — *Let V be a countable union of algebraic subvarieties of a given variety over an uncountable algebraically closed ground field. Then V admits an irredundant decomposition, and such an irredundant decomposition is unique.*

Proof. — See [1, Lemma 11]. ☐

Lemma 2.19. — *Let A be an abelian variety over k, and let K be a subgroup which can be represented as a countable union of Zariski closed subsets in A. Then the irredundant decomposition of K contains a unique irreducible component passing through* 0*, and this component is an Abelian subvariety in A.*

Proof. — See [1, Lemma 12]. ☐

Regular maps into $\mathsf{CH}_0(X)$. — Let X be a nonsingular projective variety over an uncountable algebraically closed field of characteristic zero.

***Definition 2.20* (c-closed, c-open).** — A subset of an integral algebraic scheme T which is union of a countable number of closed subsets is called a *c-closed* subset and the complement of a c-closed, i.e., intersections of a countable number of open subsets is called a *c-open* subset.

Here we work over an uncountable field because in this case the theorem on unique decomposition into irreducible components extends to c-closed subsets, so we can speak about the dimension of a c-closed subset, understanding by this the maximum of the dimensions of its irreducible components.

Definition 2.21 **(Symmetric product).** — The *d-th symmetric product* of a variety X, denoted by $\mathsf{Sym}^d(X)$, is the quotient variety $\mathsf{Sym}^d(X) = X^d/\Sigma_d$, where X^d is the self-product of X and Σ_d is the group of permutations of the factors.

The d-th symmetric product $\mathsf{Sym}^d(X)$ is a variety of dimension nd, where $n = \dim(X)$ and as a set coincides with the set of effective 0-cycles of degree d, i.e.,

$$\mathsf{Sym}^d(X) \underset{\text{as set}}{=} \{\text{effective 0-cycles of degree } d \text{ on } X\}.$$

Definition 2.22 **(Difference map).** — The set-theoretic map

$$\begin{array}{rccc} \theta^X_{d_1,d_2}: & \mathsf{Sym}^{d_1}(X)\times\mathsf{Sym}^{d_2}(X) & \longrightarrow & \mathsf{CH}_0(X) \\ & (A,B) & \longmapsto & [A-B] \end{array}$$

where $[A-B]$ is the class of the cycle $A-B$ modulo rational equivalence, will be called the *difference map.*

Remark 2.23. — When $d_1 = d_2 = d$ we will denote $\theta^X_{d_1,d_2}$ just by θ^X_d.

For any non negative integers $d_1,\dots,d_s$ we denote by

$$\mathsf{Sym}^{d_1,\dots,d_s}(X) = \mathsf{Sym}^{d_1}(X)\times\cdots\times\mathsf{Sym}^{d_s}(X).$$

to the fibred product over the ground field k.
Let

$$\begin{aligned} W^{d_1,d_2} &= \{(A,B;C,D)\in\mathsf{Sym}^{d_1,d_2}(X)\times\mathsf{Sym}^{d_1,d_2}(X) : \theta^X_{d_1,d_2}(A,B) = \theta^X_{d_1,d_2}(C,D)\} \\ &= \{(A,B;C,D)\in\mathsf{Sym}^{d_1,d_2}(X)\times\mathsf{Sym}^{d_1,d_2}(X) : (A-B)\sim_{\mathsf{rat}}(C-D)\} \end{aligned}$$

be the subset of $\mathsf{Sym}^{d_1,d_2}(X)\times\mathsf{Sym}^{d_1,d_2}(X)$ defining the rational equivalence on $\mathsf{Sym}^{d_1,d_2}(X)$. It is a c-closed subset by Lemma 1 in [20].

Remark 2.24. — Note that W^{d_1,d_2} is the fibred product $\mathsf{Sym}^{d_1,d_2}(X)\times_{\mathsf{CH}_0(X)}\mathsf{Sym}^{d_1,d_2}(X)$.

Definition 2.25 **(Regular map into $\mathsf{CH}_0(X)$).** — A set-theoretic map $\kappa: Z\to\mathsf{CH}_0(X)$ of an algebraic variety Z into the Chow group of 0-cycles $\mathsf{CH}_0(X)$ will be called *regular* if there exists a commutative diagram (in the set-theoretic sense)

$$\begin{array}{ccc} Y & \xrightarrow{f} & \mathsf{Sym}^{n,m}(X) \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle \theta^X_{d_1,d_2}} \\ Z & \xrightarrow{\kappa} & \mathsf{CH}_0(X) \end{array}$$

where f is a regular map and g is an epimorphism which is also a regular map.

Equivalently set-theoretic regular maps into $\mathsf{CH}_0(X)$ can be defined as follows ([20, Lemma 4]).

***Definition 2.26* (Alternative definition of a regular map into $\mathsf{CH}_0(X)$).** — The set-theoretic map $\kappa : Z \to \mathsf{CH}_0(X)$ is *regular* if and only if for any integers d_1 and d_2 the subset

$$W_{\kappa,\theta^X_{d_1,d_2}} = \{(z,A,B) \in Z \times \mathsf{Sym}^{d_1,d_2}(X) : \kappa(z) = \theta^X_{d_1,d_2}(A,B)\} = Z \times_{\mathsf{CH}_0(X)} \mathsf{Sym}^{n,m}(X)$$

is c-closed.

Lemma 2.27. — *The map $\theta^X_{d_1,d_2} : \mathsf{Sym}^{d_1,d_2}(X) \to \mathsf{CH}_0(X)$ is regular.*

Proof. — It follows from the above alternative definition of a regular map into $\mathsf{CH}_0(X)$ and the fact that the subset $W^{d_1,d_2} = \mathsf{Sym}^{d_1,d_2}(X) \times_{\mathsf{CH}_0(X)} \mathsf{Sym}^{d_1,d_2}(X)$ is c-closed ([20, Lemma 1]). □

Remark 2.28. — Recall that $\mathsf{CH}_0(X) = \mathbb{Z} \times \mathsf{CH}_0(X)_{\deg=0}$, where $\mathsf{CH}_0(X)_{\deg=0}$ is the Chow group of 0-cycles of degree zero, see [20, Introduction].

Lemma 2.29. — *Let $\kappa : Z \to \mathsf{CH}_0(X)_{\deg=0}$ be a regular map and let*

$$alb_X : \mathsf{CH}_0(X)_{\deg=0} \longrightarrow \mathrm{Alb}(X)$$

be the Albanese map. Then the composite map $alb_X \circ \kappa : Z \to \mathrm{Alb}(X)$ is a regular map of algebraic varieties.

Proof. — see [20, Lemma 8]. □

Representability of $\mathsf{CH}_0(X)_{\deg=0}$. — Let X be a connected smooth projective variety over $\mathbb{C}$ of dimension n.

***Definition 2.30* (Representability).** — $\mathsf{CH}_0(X)_{\deg=0}$ is *representable* if the natural map $\theta^X_d : \mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X) \to \mathsf{CH}_0(X)_{\deg=0}$ is surjective for sufficiently large d (see [23, Definition 10.6]).

Equivalently the representability of $\mathsf{CH}_0(X)_{\deg=0}$ can be defined as follows ([23, Thm. 10.11]).

***Definition 2.31* (Representability).** — The group $\mathsf{CH}_0(X)_{\deg=0}$ is *representable* if and only if $alb_X : \mathsf{CH}_0(X)_{\deg=0} \to \mathrm{Alb}(X)$ is an isomorphism.

The following Lemma is a basic property of rational equivalence ([23, Lemma 10.7], [16, Lemma 3]).

Lemma 2.32. — *The fibres of the map $\theta^X_d : \mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X) \to \mathsf{CH}_0(X)_{\deg=0}$ are countable unions of closed algebraic subsets of $\mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X)$.*

2.3. Algebraic Equivalence. —

***Definition 2.33* (Algebraic equivalence).** — Let X be a smooth projective reduced scheme. A cycle Z of codimension r on X is *algebraically equivalent to* 0, denoted by $Z \sim_{\mathsf{alg}} 0$, if and only if there exits a smooth connected curve C, a cycle $W \in \mathsf{Z}^r(X \times C)$ such that for any $t \in C$ defining by

$$W(t) := (\mathsf{pr}_X)_*(W \cdot (X \times \{t\}))$$

we have: $W(t_1) = Z$ and $W(t_2) = 0$ for some $t_1, t_2 \in C$.

The set of cycles of codimension r that are algebraically equivalent to 0 is denoted by $\mathsf{Z}^r(X)_{\mathsf{alg}} = \{Z \in \mathsf{Z}^r(X) : Z \sim_{\mathsf{alg}} 0\}$. It is a subgroup of $\mathsf{Z}^r(X)$.
If X is a complex smooth projective variety equivalently we can define algebraic equivalence as follows, see [23, §8.2.1], [4, Introduction].

***Definition 2.34* (Cycle associated to an intersection).** — Let C be a smooth connected curve and $W \subset C \times X$ a closed algebraic subset of codimension r of X each of whose components dominates C (i.e. the restriction to the components of the projection over C is dominant). Then, for each point $t \in C$ we can consider: $W(t) = [W \cap (\{t\} \times X)]$, the cycle of codimension r on X associated to the schematic intersection of W with the fiber $\{t\} \times X \simeq X$ over t via the first projection.

Remark 2.35. — Note that if we denote by $\tau = \mathsf{pr}_X|_W : W \to X$ and by $\pi = \mathsf{pr}_C|_W : W \to C$, where pr_X and pr_C are the projections to X and C respectively, we can define $W(t)$ as follows: $W(t) = \tau_* \circ \pi^*(t)$.

***Definition 2.36* (Alternative definition of $\mathsf{Z}^r(X)_{\mathsf{alg}}$).** — The subgroup $\mathsf{Z}^r(X)_{\mathsf{alg}}$ of cycles of codimension r algebraically equivalent to 0 is the subgroup generated by the cycles of codimension r the form $W(t) - W(t')$, for any smooth connected curve C, any points $t, t' \in C$, and for any cycle $W \in \mathsf{Z}^r(C \times X)$ each of whose components dominates C.

The group $A_r(X)$. — By definition it is clear that $\mathsf{Z}_r(X)_{\mathsf{rat}} \subset \mathsf{Z}_r(X)_{\mathsf{alg}}$, so we can define the following group quotient

***Definition 2.37* (The continuous part of the Chow group).** — The group quotient of cycles algebraically equivalent to 0 modulo rational equivalence is denoted by

$$\mathsf{A}_r(X) = \frac{\mathsf{Z}_r(X)_{\mathsf{alg}}}{\mathsf{Z}_r(X)_{\mathsf{rat}}} \subset \mathsf{CH}_r(X).$$

This group should be thought of as the continuous part of the Chow group of r-cycles (see [3, Introduction]).

2.4. Homological Equivalence. — Let X be a smooth projective variety over an algebraically closed field k.

***Definition 2.38* (Weil-cohomology).** — Fix a field F of characteristic 0 called the coefficient field. A *Weil-cohomology theory* is a contravariant functor $X \to H^*(X)$ from the category of varieties to the category of augmented, finite dimensional, anti-commutative F-algebras which satisfies the following properties (see [12, §1.2.])

1. Poincaré duality: if $\dim(X) = n$, then
   (a) The groups $H^r(X) = 0$, for $r \neq 0, \ldots, 2n$.
   (b) There is a given orientation isomorphism: $H^{2n}(X) \simeq F$ (note that in particular, $H^0(P) \simeq F$, for P a point).
   (c) The canonical pairings: $H^r(X) \times H^{2n-r}(X) \to H^{2n}(X)$ are non-singular.

   Let $H_r(X)$ be the F-vector space dual to $H^r(X)$. Then Poincaré duality states that there are isomorphisms: $H^{2n-r}(X) \xrightarrow{\sim} H_r(X)$ induced by the map $a \mapsto \langle \cdot, a\rangle$, where $\langle \cdot, \cdot \rangle : H^*(X) \to F$ is the degree map.

Let $f : X \to Y$ be a morphism, $f^* = H^*(f) : H^*(Y) \to H^*(X)$ and define a F-linear map $f_* : H^*(X) \to H^*(Y)$ as the transpose of f^*. Then $f_*((f^*a) \cdot b) = a \cdot f_*b$.

2. Künneth formula: let $\mathsf{pr}_X : X \times Y \to X$ and $\mathsf{pr}_Y : X \times Y \to Y$ be the projections. Then

$$\begin{array}{ccc} H(X) \otimes_F H(Y) & \longrightarrow & H(X \times Y) \\ a \otimes b & \longmapsto & \mathsf{pr}_X^*(a) \cdot \mathsf{pr}_Y^*(b) \end{array}$$

is an isomorphism.

3. Cycle maps: there are groups homomorphisms $cl_X : \mathsf{Z}^r(X) \to H^{2r}(X)$ satisfying the following properties:

   - functorial in the sense that for a morphism of varieties $f : X \to Y$, one has $f^* \circ cl_Y = cl_X \circ f^*$ and $f_* \circ cl_X = cl_Y \circ f_*$.
   - Multiplicativity: $cl_{X \times Y}(Z \times W) = cl_X(Z) \otimes cl_Y(W)$;
   - Non-triviality: if P is a point, then $cl : \mathsf{Z}^*(P) = \mathbb{Z} \to H^*(P) = F$ is the canonical inclusion.

The elements of $H^*(X)$ are called *cohomology classes*, the multiplication on $H^*(X)$ is called *cup product*.

Remark 2.39. — There are other more restrictive definitions of a Weil cohomology theory, see for example [18, Definition 1.2.13], where they define the Weil cohomology over the category of smooth projective reduced schemes over an arbitrary field k and they state that a Weil cohomology theory also satisfies the following properties:

a. There are *cycle class* maps: $\mathrm{cl}_X : \mathsf{CH}^r(X) \to H^{2r}(X)$ which are functorial, compatible with intersection product and compatible with points.

b. Weak Lefschetz holds: if $i : Y \hookrightarrow X$ is a smooth hyperplane section of a variety of dimension n, then $H^r(X) \xrightarrow{i^*} H^r(Y)$ is an isomorphism for $r < n-1$ and is injective for $r = n-1$.

c. Hard Lefschetz holds: the Lefschetz operator $L(\alpha)$ induces isomorphisms

$$L^{n-r} : H^{n-r}(X) \xrightarrow{\sim} H^{n+r}(X), 0 \leq r \leq n.$$

Fixing a Weil-cohomology theory we can now define the homological equivalence

***Definition 2.40* (Homological equivalence).** — A cycle Z of codimension r on X is *homologically equivalent to* 0, denoted by $Z \sim_{\mathrm{hom}} 0$, if $cl_X(Z) = 0$.

This definition depends on the choice of a Weil cohomology theory ([18, Definition 1.2.16]). The set of cycles of codimension r homologically equivalent to 0 form a group, it is denoted by $\mathsf{Z}^r(X)_{\mathrm{hom}}$.

The group $\mathsf{CH}_r(X)_{\mathrm{hom}}$. — Let X be a smooth complex quasi-projective variety or more generally let X be a smooth projective reduced scheme over an arbitrary field k, fixing a Weil cohomology theory we have the following lemma

Lemma 2.41. — *Let* $\dim(X) = n$ *and let* $cl_X(Z) \in H^{2n-2r}(X)$ *be the class of a* r*-cycle* Z *on* X*. If* $Z \sim_{\mathsf{rat}} 0$*, then* $cl_X(Z) = 0$*.*

Proof. — For the complex case, see proof of [23, Lemma 9.18]. For the general case, note that this property forms part of the properties of the Weil cohomology theory (see item a of Remark 2.39). □

By this lemma and the fundamental theorem on homomorphisms, the cycle map: $cl_X : \mathsf{Z}_r(X) \to H^{2n-2r}(X)$ thus gives the *cycle class map*

$$\mathrm{cl}_X : \mathsf{CH}_r(X) \longrightarrow H^{2n-2r}(X).$$

Definition 2.42 **(The group $\mathsf{CH}_r(X)_{\mathrm{hom}}$).** — The kernel of the cycle *class* map cl_X is denoted by

$$\mathsf{CH}_r(X)_{\mathrm{hom}} = \mathsf{Ker}(\mathrm{cl}_X : \mathsf{CH}_r(X) \longrightarrow H^{2n-2r}(X))$$

Note that $\mathsf{CH}_r(X)_{\mathrm{hom}}$ is the group of r-cycles homologically equivalent to 0 modulo rational equivalence, i.e., $\mathsf{CH}_r(X)_{\mathrm{hom}} = \frac{\mathsf{Z}_r(X)_{\mathrm{hom}}}{\mathsf{Z}_r(X)_{\mathsf{rat}}}$.

2.5. Relation between algebraic, rational and homological equivalence. — Let X be a smooth projective reduced scheme over an algebraically closed field k of characteristic 0, then we have the following proposition.

Proposition 2.43. — $\mathsf{Z}^r(X)_{\mathsf{rat}} \subset \mathsf{Z}^r(X)_{\mathsf{alg}} \subset \mathsf{Z}^r(X)_{\mathrm{hom}}$.

Proof. — The inclusion $\mathsf{Z}^r(X)_{\mathsf{rat}} \subset \mathsf{Z}^r(X)_{\mathsf{alg}}$ is clear by definition.
To prove the second inclusion $\mathsf{Z}^r(X)_{\mathsf{alg}} \subset \mathsf{Z}^r(X)_{\mathrm{hom}}$, assume that $Z \in \mathsf{Z}^r(X)_{\mathsf{alg}}$, that is, $Z \sim_{\mathsf{alg}} 0$, then by definition of algebraic equivalence there exits a connected smooth curve C, a cycle $W \in \mathsf{Z}^r(X \times C)$ each of whose components dominates C and points $t_1, t_2 \in C$ such that Z is of the form $W(t_1) - W(t_1)$, where $W(t_1) = \tau_* \circ \pi^*(t_1)$ and $W(t_2) = \tau_* \circ \pi^*(t_2)$ with τ (resp. π) the restriction to W of the projection to X (resp. to C), see Remark 2.35.
Now consider the following commutative diagram

$$\begin{array}{ccc}
\mathsf{Z}^r(X) & \xrightarrow{cl} & H^{2r}(X) \\
{\scriptstyle \tau_*}\uparrow & & {\scriptstyle \tau_*}\uparrow \\
\mathsf{Z}^1(W) & \xrightarrow{cl} & H^2(W) \\
{\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi^*}\uparrow \\
\mathsf{Z}^1(C) & \xrightarrow{cl} & H^2(C)
\end{array}$$

Note that since C is connected the cycle map $cl : \mathsf{Z}^1(C) \to H^2(C) \simeq \mathbb{Z}$ coincides with the degree map $\deg : \mathsf{Z}_0(C) \to \mathbb{Z}$ of 0-cycles on C, so for $t_1, t_2 \in C$ we have $cl(t_1) = cl(t_2) = 1$, then

$$\tau_* \circ \pi^* \circ cl(t_1) = \tau_* \circ \pi^* \circ cl(t_2).$$

By the commutativity of the diagram we have: $cl \circ \tau_* \circ \pi^*(t_1) = \tau_* \circ \pi^* \circ cl(t_1)$ and analogously we have $\tau_* \circ \pi^* \circ cl(t_2) = cl \circ \tau_* \circ \pi^*(t_2)$, it follows that $cl \circ \tau_* \circ \pi^*(t_1) = cl \circ \tau_* \circ \pi^*(t_2)$ which is equivalent to $cl(W(t_1)) = cl(W(t_2))$. Since the cycle map is an homomorphism we have $cl(W(t_1) - W(t_2)) = cl(Z) = 0$, then $Z \sim_{\mathrm{hom}} 0$, that is, $Z \in \mathsf{Z}^r(X)_{\mathrm{hom}}$. □

The above proposition is also true over an arbitrary field, see [18, §1.2.1.].

Rational, algebraic and homological equivalence of 0*-cycles.* — Let X be a smooth projective reduced scheme of dimension n over an algebraically closed field k of characteristic 0.
From the Proposition 2.43 for the case $r = n$, one has $\mathsf{Z}^n(X)_{\mathsf{alg}} \subset \mathsf{Z}^n(X)_{\mathrm{hom}}$ or equivalently $\mathsf{Z}_0(X)_{\mathsf{alg}} \subset \mathsf{Z}_0(X)_{\mathrm{hom}}$.
Now in this subsection we will prove that if in addition X is connected the other inclusion also holds, therefore $\mathsf{Z}_0(X)_{\mathrm{hom}} = \mathsf{Z}_0(X)_{\mathsf{alg}}$. To prove it we use the following lemmas

Lemma 2.44. — *(0-cycles homologous to 0 have degree 0 and vice versa) Assume in addition that X is connected, and let $\mathsf{Z}_0(X)_{\mathrm{deg}=0} \subset \mathsf{Z}_0(X)$ be the group of 0-cycles of degree 0. Then $\mathsf{Z}_0(X)_{\mathrm{hom}} = \mathsf{Z}_0(X)_{\mathrm{deg}=0}$.*

Proof. — Since X is connected we have $H^{2n}(X) \simeq \mathbb{Z}$. Then

$$\mathsf{Z}_0(X)_{\mathrm{hom}} = \mathsf{Ker}(cl : \mathsf{Z}_0(X) \longrightarrow H^{2n}(X)) = \mathsf{Ker}(\deg : \mathsf{Z}_0(X) \longrightarrow \mathbb{Z}) = \mathsf{Z}_0(X)_{\mathrm{deg}=0},$$

i.e. the 0-cycles homologous to 0 coincide with the 0 cycles of degree 0 □

Lemma 2.45. — *If C be an integral (reduced and irreducible) smooth curve and $P_1, P_2 \in C$ two points in C, then $P_1 \sim_{\mathsf{alg}} P_2$.*

Proof. — We must show that there exists a smooth curve D, $W \in \mathsf{Z}^1(C \times D)$ i.e. a family of 0-cycles (or equivalently cycles of codimension 1) on C each of whose components dominates D, and points $t_1, t_2 \in D$ such that $W(t_1) = P_1$ and $W(t_2) = P_2$, i.e., such that P_1 and P_2 are members of this family.
It is enough to take $D = C$, $W = \Delta = \{(a, b) \in C \times C : a = b\} \subset C \times C$ the diagonal, and $t_1 = P_1$ and $t_1 = P_2$. □

Lemma 2.46. — *If there exists a connected curve C such that its components are smooth and integral and P and Q are two points of C, then $P \sim_{\mathsf{alg}} Q$.*

Proof. — Assume that such a curve $C = C_1 \cup \cdots \cup C_r$ exists, then without lost of generality we can assume that $P = P_0 \in C_1$, $Q = P_r \in C_r$ and that $P_i \in C_i \cap C_{i+1}$ for all $i = 1, \ldots, r-1$, then for the Lemma 2.45 we have $P_{i-1} \sim_{\mathsf{alg}} P_i$ for all $i = 1, \ldots, r$, then $P = P_0 \sim_{\mathsf{alg}} Q = P_r$. □

***Proposition 2.47* (Homological and algebraic equivalence coincide for 0-cycles).** — *If X is connected, then $\mathsf{Z}_0(X)_{\mathrm{hom}} = \mathsf{Z}_0(X)_{\mathsf{alg}}$.*

Proof. — By Proposition 2.43 it is enough to prove that $\mathsf{Z}_0(X)_{\mathrm{hom}} \subset \mathsf{Z}_0(X)_{\mathsf{alg}}$. By Lemma 2.44 this is equivalent to prove that if a 0-cycle has degree zero then it is algebraically equivalent to 0, which we do next.
Let $Z = \sum m_i P_i$ be a 0-cycle on X with degree zero, i.e., with $\sum m_i = 0$, then we can consider $Z = P - Q$ with $P, Q \in X$, this holds true because since Z has degree zero it is generated by differences of this form. Indeed, observe that $Z = \sum m_i P_i = \sum m_i (P_i - Q) = \sum m_i P_i - \sum m_i Q$, for any $Q \in X$. By [[23], §8.2.1] there exits a smooth connected curve C containing $P, Q \in X$, then by Lemma 2.46 $P \sim_{\mathsf{alg}} Q$ or equivalently $Z = P - Q \sim_{\mathsf{alg}} 0$. □

Finally, we next study the relation between rational, algebraic and homological equivalence of 0-Cycles on a smooth projective variety.

Proposition 2.48. — *Let X be a smooth projective variety of dimension n over an algebraically closed field k of characteristic* 0. *Then* $\mathsf{CH}_0(X)_{\mathrm{deg}=0} = \mathsf{CH}_0(X)_{\mathrm{hom}} = \mathsf{A}_0(X)$.

Proof. — Since by Lemma 2.44 we have $\mathsf{Z}_0(X)_{\mathrm{deg}=0} = \mathsf{Z}_0(X)_{\mathrm{hom}}$ the first equality holds. On the other hand, by Proposition 2.47 we have that $\mathsf{Z}_0(X)_{\mathrm{hom}} = \mathsf{Z}_0(X)_{\mathsf{alg}}$, this gives the second equality. □

If $\dim(X) = 1$, that is, X is a smooth projective curve over an algebraically closed field k of characteristic 0, we get the following important lemma for the proof of the main result of this thesis.

***Lemma 2.49* (Fact 2).** — *Let C be a smooth projective curve over an algebraically closed field k of characteristic* 0. *Then* $\mathsf{CH}_0(C)_{\mathrm{deg}=0} = \mathsf{CH}_0(C)_{\mathrm{hom}} = \mathsf{A}_0(C)$.

Proof. — Apply Proposition 2.48 when $\dim(X) = 1$. □

3. Hodge Theory and The Abel–Jacobi Map

In this section we recall some facts about Hodge theory. We start with the notion of Hodge structure and polarized varieties, then we study the notion of complex torus associated to the Hodge structure on the cohomology group $H^1(X)$ of a compact Kähler manifold X showing that it coincides with the Picard group $\mathrm{Pic}^0(X)$ and that if X is smooth and projective $\mathrm{Pic}^0(X)$ is an abelian variety. Next we define the morphism of Hodge structures.
In this section we also define the k-th intermediate Jacobian which is a complex torus associated to the $(2k-1)$-th cohomology group of X, then we see that the Jacobian torus $J(X)$ coincides with $\mathrm{Pic}^0(X)$ so if X is smooth and projective it has the structure of an abelian variety, we next study the relation between the cohomology group of a curve and its Jacobian, after that we define the Albanese map and the Albanese variety. In particular, the topics of this section gives us the background to prove *Fact 3* (Lemma 3.33) and *Fact 1* (Lemma 3.46). The main reference for this section is [22], see also [19].

3.1. Hodge Structure and Polarized Varieties.

— First recall the definition of a Kähler manifold.

***Definition 3.1* (Kähler manifold).** — A *Kähler manifold* is a complex manifold equipped with a hermitian metric whose imaginary part, which is a 2-form of type $(1,1)$ relative to the complex structure, is closed. This 2-form is called the *Kähler form*.

Example 3.2. — Smooth projective complex manifolds are special cases of compact Kähler manifolds.

Hodge structure. —

***Definition 3.3* (Integral Hodge structure of weight k).** — An *integral Hodge structure of weight k* is given by a free abelian group $V_{\mathbb{Z}}$ of finite type, together with a decomposition of its complexification: $V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$ such that $V^{p,q} = \overline{V^{q,p}}$.

***Definition 3.4* (Integral Hodge substructure).** — An *integral Hodge substructure* is a subgroup $W_{\mathbb{Z}} \subset V_{\mathbb{Z}}$ such that $W_{\mathbb{C}} := W_{\mathbb{Z}} \otimes \mathbb{C}$ has a decomposition induced by that of $V_{\mathbb{C}}$, i.e., $W_{\mathbb{C}} = \bigoplus_{p+q=k} W_{\mathbb{C}} \cap V^{p,q}$.

Example 3.5. — The integral cohomology group $H^k(X, \mathbb{Z})$ of a compact Kähler manifold carries a weight k Hodge structure. Indeed, recall that given a compact complex manifold X, there is an isomorphism: $\mathcal{H}^k(X) \cong H^k(X, \mathbb{C})$, where $\mathcal{H}^k(X)$ is the set of complex valued harmonic forms for the Laplacian associated to any metric on X. When the metric is Kähler there is a decomposition of harmonic forms into harmonic forms of type (p, q). Thus, there is an induced decomposition: $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$, where $H^{p,q}(X)$ is the set of classes representable by closed forms of type (p, q) and it satisfies the Hodge symmetry: $H^{p,q}(X) = \overline{H^{q,p}(X)}$. This decomposition is called the Hodge decomposition of the cohomology of a compact Kähler manifold ([22, §6.1.3]).

Given such a decomposition, we define the *associated Hodge filtration $F^{\bullet}V$* by: $F^pV_{\mathbb{C}} = \bigoplus_{r \geq p} V^{r,k-r} = V^{p,k-p} \oplus \cdots \oplus V^{k,0}$. It is a decreasing filtration on $V_{\mathbb{C}}$, which satisfies: $V_{\mathbb{C}} = F^pV_{\mathbb{C}} \oplus \overline{F^{k-p+1}V_{\mathbb{C}}}$.

Remark 3.6. — Hodge filtration determines the Hodge decomposition by $V^{p,q} = F^pV_{\mathbb{C}} \cap \overline{F^qV_{\mathbb{C}}}$.

Let X be an n-dimensional compact Kähler manifold of Kähler form ω, then the cup product with the class $[\omega] \in H^2(X, \mathbb{R})$ of ω gives the Lefschetz operator

$$L : H^k(X, \mathbb{R}) \longrightarrow H^{k+2}(X, \mathbb{R}).$$

This operator gives:

- the *Lefschetz decomposition*: $H^k(X, \mathbb{R}) = \bigoplus_r L^r H^{k-2r}_{prim}$, where each component admits an induced Hodge decomposition,
- an *intersection form* on $H^k(X, \mathbb{R})$ for $k \leq n$: $Q(\alpha, \beta) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta = \langle L^{n-k}\alpha, \beta \rangle$. Q is alternating if k is odd, symmetric otherwise.

Then we have an Hermitian form: $H(\alpha, \beta) = i^k Q(\alpha, \bar{\beta})$ on $H^k(X, \mathbb{C})$, induced by the intersection form Q.

Definition 3.7. — The class $[\omega]$ of the Kähler form ω of X is called *integral* if $[\omega]$ belongs to $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$.

Definition 3.8. — (Polarized Hodge structure) An *integral polarised Hodge structure of weight k* is given by a Hodge structure $(V_{\mathbb{Z}}, F^pV_{\mathbb{C}})$ of weight k, together with an intersection form Q on $V_{\mathbb{Z}}$, which is symmetric if k is even, alternating otherwise, and satisfies conditions (i) and (ii) in [22, §7.1.2].

Polarized varieties. —

***Definition 3.9* (Polarized manifold).** — A *polarized manifold* is a pair $(X, [\omega])$, where X is a compact complex manifold, and $[\omega]$ is an integral Kähler class on X.

***Definition 3.10* (Chern form).** — Let X be a complex manifold, $\mathcal{L}$ a holomorphic line bundle on X, and h a Hermitian metric on $\mathcal{L}$. The 2-form $\omega_{\mathcal{L},h}$, which is closed and real of type $(1,1)$, is called the *Chern form* associated to the hermitian metric h on $\mathcal{L}$. We say that $\omega_{\mathcal{L},h}$ is *positive* if it correspond to a Hermitian metric on X ([22, 3.3.1]).

As a consequence of Theorem 7.10 in [22], given a polarised manifold $(X, [\omega])$ there exits a holomorphic line bundle $\mathcal{L}$ on X and a Hermitian metric h such that $\omega_{\mathcal{L},h} = \omega$ is a positive form. We say that $\mathcal{L}$ is positive and we have the following theorem called Kodaira Embedding Theorem.

***Theorem 3.11* (Kodaira Embedding Theorem).** — *Let X be a compact complex manifold and $\mathcal{L}$ a positive holomorphic line bundle on X. Then for every sufficiently large $N \in \mathbb{Z}$ there exists a holomorphic embedding $\phi : X \to \mathbb{P}^r$ such that $\phi^*(\mathcal{O}_{\mathbb{P}^r}(1)) = \mathscr{L}^{\otimes N}$, where $\mathscr{L}$ is the sheaf of holomorphic sections of $\mathcal{L}$.*

Proof. — See [22, Theorem 7.11]. □

Corollary 3.12. — *A polarized manifold is a projective variety, i.e., admits a holomorphic embedding into a projective space.*

Proof. — It follows from the comment above the Kodaira Embedding Theorem together with the Kodaira Embedding Theorem. □

Abelian variety associated to the Hodge structure of weight 1. — Let X be compact Kähler manifold. The Hodge structure on $H^1(X)$ is described by the decomposition: $H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$, satisfying $H^{0,1}(X) = \overline{H^{1,0}(X)}$.
Then we have an isomorphism of real vector spaces $H^1(X, \mathbb{R}) \to H^{0,1}(X)$ obtained by the composition of the inclusion $H^1(X, \mathbb{R}) \subset H^1(X, \mathbb{C})$ and the projection $H^1(X, \mathbb{C}) \to H^{0,1}(X)$ given by the Hodge decomposition of $H^1(X, \mathbb{C})$.
It follows that the lattice $H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R})$ projects onto a lattice in the complex vector space $H^{0,1}(X)$. Thus identifying this last lattice with $H^1(X, \mathbb{Z})$ via the above isomorphism we have a complex torus

$$T = \frac{H^{0,1}(X)}{H^1(X, \mathbb{Z})}$$

associated to the Hodge structure on $H^1(X)$.
Now we prove that the complex torus T coincides with the group of isomorphism classes of holomorphic line bundles over X with Chern class 0.
For this recall that we have the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$, then we have an associated long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathbb{Z}) &\xrightarrow{\psi_0} H^0(X, \mathcal{O}_X) \xrightarrow{\varphi_0} H^0(X, \mathcal{O}_X^*) \xrightarrow{c_0} H^1(X, \mathbb{Z}) \xrightarrow{\psi_1} H^1(X, \mathcal{O}_X) \\ &\xrightarrow{\varphi_1} H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X^*) \xrightarrow{c_2} \dots \end{aligned}$$

Recall also that the group $H^1(X, \mathcal{O}_X^*)$ can be identified with the Picard group of isomorphism classes of holomorphic line bundles L over X (see [22, Theorem 4.49]).

***Definition 3.13* (First Chern class homomorphism).** — The connecting homomorphism $c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ is called the *first Chern class homomorphism* and the class $c_1(L)$ is called the *first Chern class of L*, see [22, p. 162].

***Definition 3.14* (The group $\mathrm{Pic}^0(X)$).** — Set

$$\mathrm{Pic}^0(X) = \mathsf{Ker}(c_1 : H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z})),$$

for the kernel of the first Chern class homomorphism c_1. This is the group of the isomorphism classes of holomorphic line bundles over X of first Chern class zero.

Proposition 3.15. — *$T = \mathrm{Pic}^0(X)$.*

Proof. — From the exactness of the above long sequence we have $\mathrm{Pic}^0(X) = \mathrm{im}(\varphi_1)$. By the fundamental theorem of homomorphisms we have $\mathrm{im}(\varphi_1) \cong \frac{H^1(X,\mathcal{O}_X)}{\mathsf{Ker}(\varphi_1)}$ and by the exactness of the above long sequence again we have that $\mathsf{Ker}(\varphi_1) = \mathrm{im}(\psi_1)$, then $\mathrm{Pic}^0(X) = \frac{H^1(X,\mathcal{O}_X)}{\mathrm{im}(\psi_1)}$. By [22, §7.2.2] we have a natural isomorphism $H^{0,1}(X) \cong H^1(X, \mathcal{O}_X)$, so $\mathrm{Pic}^0(X) = \frac{H^{0,1}(X)}{\mathrm{im}(\psi_1)}$. So, in order to prove the proposition it is enough to prove that we can identify $\mathrm{im}(\psi_1) = \psi_1(H^1(X, \mathbb{Z}))$ with $H^1(X, \mathbb{Z})$ itself, that is, we must prove that ψ_1 is injective so $H^1(X, \mathbb{Z})$ is really isomorphic to $\psi_1(H^1(X, \mathbb{Z}))$. Indeed, by the exactness of the above long sequence we have $\mathsf{Ker}(\psi_1) = \mathrm{im}(c_0)$, so in order to prove that ψ_1 is injective we must prove that $\mathsf{Ker}(\psi_1) = \mathrm{im}(c_0) = 0$, i.e., that c_0 is the zero map or in other words that $\mathsf{Ker}(c_0) = H^0(X, \mathcal{O}_X^*)$. But since $\mathsf{Ker}(c_0) = \mathrm{im}(\varphi_0)$, by the exactness of the above long sequence, it is enough to prove that $\mathrm{im}(\varphi_0) = H^0(X, \mathcal{O}_X^*)$, i.e., that φ_0 is surjective. Assuming that X is projective we have $H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$ and φ_0 is the exponential map of complex numbers so it is surjective. □

Remark 3.16. — The torus T associated to the Hodge structure on $H^1(X)$ is itself a Kähler manifold, and thus admits a Hodge structure on its group $H^1(T)$, see [22, p. 169].

Now let us study the relationship between the Hodge structure on $H^1(X)$ and the Hodge structure on $H^1(T)$ (see [22, §7.2.1]).

Lemma 3.17. — *The Hodge structure on $H^1(T, \mathbb{Z})$ is dual to that of $H^1(X, \mathbb{Z})$.*

Proof. — For a torus $T = \frac{V}{\Gamma}$, where V is a complex vector space, we have a natural identification: $\Gamma = H_1(T, \mathbb{Z})$ and $V^* = H^1(T, \mathbb{R})$. Furthermore, the holomorphic cotangent bundle of T is trivial, as its global sections are given by the complex linear forms on V, considered as holomorphic forms on V invariant under Γ. It follows that $H^{1,0}(T) = V^*$. Thus the Hodge structure on $H^1(T, \mathbb{Z})$ is dual to that of $H^1(X, \mathbb{Z})$, that is, $H^1(T, \mathbb{Z}) = H^1(X, \mathbb{Z})^*$ and $H^{1,0}(T) = H^{0,1}(X)^*$. □

In what follows we prove the following proposition (see [22, Proposition 7.16])

Proposition 3.18. — *The complex torus $T = \mathrm{Pic}^0(X)$ of a projective smooth variety is an algebraic projective variety.*

Proof. — Suppose now that X is a polarised manifold, and let L be the Lefschetz operator acting on the integral cohomology of X. Obviously, the cohomology of degree 1 is primitive, and thus the alternating intersection form $Q(\alpha, \beta) = \langle L^{n-1}\alpha, \beta\rangle$, $n = \dim(X)$, defined on $H^1(X)$ and with integral values on $H^1(X, \mathbb{Z})$ satisfies the property that the Hermitian form

$H(\alpha,\beta) = iQ(\alpha,\bar{\beta})$ is positive definite on $H^{1,0}(X)$, which is orthogonal to $H^{0,1}(X)$ for H, equivalently, this means that the form $Q \in \bigwedge^2(H^1(X,\mathbb{Z}))^*$ can be considered as an element ω of $\bigwedge^2(H^1(T,\mathbb{Z})^*) = \bigwedge^2(H^1(T,\mathbb{Z})) = H^2(T,\mathbb{Z})$. In fact, the de Rham class of ω is simply the class of the constant 2-form Ω on T obtained by extending Q by $\mathbb{R}$-linearity. If we identify $H_1(T,\mathbb{Z})$ with $H^1(X,\mathbb{Z})$, and thus $H_1(T,\mathbb{Z}) \otimes \mathbb{R}$ with $H^1(X,\mathbb{R})$, this differential form $\Omega = Q$ on $H^1(X,\mathbb{R})$.
The properties of Q then imply that the form Ω is a Kähler form on T. As the Kähler form thus defined on T is a of integral class, T is polarized manifold and Kodaira's theorem (Theorem 3.11) implies that T is an algebraic projective variety, see also Corollary 3.12. □

Definition 3.19. — (Abelian variety) A complex torus that is also an algebraic projective variety is called an *abelian variety.*

From the Proposition 3.18 we have that the complex torus $T = \mathrm{Pic}^0(X)$ of a projective smooth variety X is an abelian variety.

Definition 3.20. — (Picard variety) The Abelian variety $T = \mathrm{Pic}^0(X)$ of a smooth projective variety X is called the *Picard variety* of X.

3.2. Morphisms of Hodge Structures. — Let $(V_{\mathbb{Z}}, F^pV_{\mathbb{C}})$ and $(W_{\mathbb{Z}}, F^pW_{\mathbb{C}})$ be Hodge structures of weight n and $m = n + 2r$, $r \in \mathbb{Z}$ respectively.

***Definition 3.21* (Morphism of Hodge structures).** — A morphism of groups $\phi : V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ is a *morphism of Hodge structures (of type* (r,r)*)* if the morphism $\phi : V_{\mathbb{C}} \to W_{\mathbb{C}}$ obtained by $\mathbb{C}$-linear extension, satisfies $\phi(F^pV_{\mathbb{C}}) \subset F^{p+r}W_{\mathbb{C}}$, or equivalently, $\phi(V^{p,q}) \subset W^{p+r,q+r}$.

Remark 3.22. — A morphism of Hodge structures ϕ induces a Hodge structure on $\mathsf{Ker}(\phi)$, see [22, Lemma 7.25].

Two important examples of morphisms of Hodge structures are the following

***Definition 3.23* (Pullback homomorphism).** — Let $\phi : X \to Y$ be a continuous map between two topological spaces. The homomorphism $\phi^* : H^k(Y,\mathbb{Z}) \to H^k(X,\mathbb{Z})$, of cohomology groups is called the *pullback homomorphism* induced by ϕ.

The pullback homomorphism is induced by the natural morphism of sheaves $\mathbb{Z}_Y \to \phi_*\mathbb{Z}_X$. There are other ways to define ϕ^*, see for example [22, §7.3.2].

Proposition 3.24. — *If $\phi : X \to Y$ is a holomorphic map between Kähler manifolds then $\phi^* : H^k(Y,\mathbb{Z}) \to H^k(X,\mathbb{Z})$ is a morphism of Hodge structures.*

Proof. — See [19, Corollary 1.13.] or [22, p. 177]. □

***Definition 3.25* (Gysin homomorphism).** — Let $\phi : X \to Y$ be a morphism between two Kähler manifolds of dimension n and $n + r$ respectively. The homomorphism in cohomology groups $\phi_* : H^k(X,\mathbb{Z}) \to H^{k+2r}(Y,\mathbb{Z})$ is called the *Gysin homomorphism* induced by ϕ.

The Gysin homomorphism is defined using Poincaré duality for X and Y as the morphism $\phi_* : H_{2n-k}(X,\mathbb{Z}) \to H_{2n-k}(Y,\mathbb{Z})$ on singular homology groups, and ϕ_* is defined in the singular chains. For other ways to define ϕ_*, see [22, §7.3.2].

Proposition 3.26. — *The Gysin morphism ϕ_* is a morphism of Hodge structures of bidegree (r,r), that is, it takes classes α of type (p,q) to classes $\phi_*(\alpha)$ of type $(p+r,q+r)$.*

Proof. — See [22, p. 179]. □

A important property for us is the following

Proposition 3.27. — *The Gysin homomorphism ϕ_* on cohomology groups induce a Hodge structure on its kernel.*

Proof. — It follows by Remark 3.22 since ϕ_* is a morphism of Hodge structures by Proposition 3.26. □

3.3. The Intermediate Jacobian. —

The k-th intermediate Jacobian. — Let X be a compact Kähler manifold.
Recall that for every $k > 0$, the Hodge filtration on $H^{2k-1}(X,\mathbb{C})$ determines the Hodge decomposition (Remark 3.6), that is, $H^{2k-1}(X,\mathbb{C}) = F^kH^{2k-1}(X) \oplus \overline{F^kH^{2k-1}(X)}$, then $F^kH^{2k-1}(X) \cap H^{2k-1}(X,\mathbb{R}) = \{0\}$, and the decomposition map gives an isomorphism: $H^{2k-1}(X,\mathbb{R}) \to \frac{H^{2k-1}(X,\mathbb{C})}{F^kH^{2k-1}(X)}$.
In consequence, the lattice $H^{2k-1}(X,\mathbb{Z})$ in $H^{2k-1}(X,\mathbb{R})$ gives a lattice in the $\mathbb{C}$-vector space $\frac{H^{2k-1}(X,\mathbb{C})}{F^kH^{2k-1}(X)}$. Identifying this last lattice with $H^{2k-1}(X,\mathbb{Z})$ via the above isomorphism we define the k-th intermediate Jacobian of a compact Kähler manifold as follows

***Definition 3.28* (The k-th intermediate Jacobian).** — The *k-th intermediate Jacobian* is defined by $J^{2k-1}(X) = \frac{H^{2k-1}(X,\mathbb{C})}{F^kH^{2k-1}(X)\oplus H^{2k-1}(X,\mathbb{Z})}$.

More generally, we can define a complex torus for every Hodge structure of weight $2k-1$ as follows (see [22, Remark 12.3])

***Definition 3.29* (Complex torus for Hodge structure of weight $2k-1$).** — Let $V_{\mathbb{Z}}$ be a Hodge structure of weight $2k-1$. The complex torus associated to it is defined by $J^{2k-1}(V) := \frac{V_{\mathbb{C}}}{(F^kV\oplus V_{\mathbb{Z}})}$.
This construction is functorial, in the sense that every morphism of Hodge structures $(V_{\mathbb{Z}}, F^\bullet V) \to (W_{\mathbb{Z}}, F^{\bullet+r}W)$ of bidegree (r,r) induces a morphism of complex tori $J^{2k-1}(V) \to J^{2(k+r)-1}(W)$.

The Jacobian of a smooth projective variety is an Abelian variety. — Recall that given a compact Kähler manifold X we define the group

$$\mathrm{Pic}^0(X) := \mathsf{Ker}(H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}))$$

(Definition 3.14) and using the long exact sequence associated to the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ we proved that $\mathrm{Pic}^0(X) = T$, where $T = \frac{H^{0,1}(X)}{H^1(X,\mathbb{Z})}$ is the complex torus associated to the Hodge structure on $H^1(X,\mathbb{Z})$ (see Proposition 3.15).
The following proposition gives an alternative definition of the complex torus $\mathrm{Pic}^0(X)$.

Proposition 3.30. — *$J^1(X) = \mathrm{Pic}^0(X)$, where $J^1(X)$ is the 1-th intermediate Jacobian of X.*

Proof. — By definition the 1-th intermediate Jacobian is $J^1(X) = \frac{H^1(X,\mathbb{C})}{F^1H^1(X)\oplus H^1(X,\mathbb{Z})}$. Now recall that there is an identification $\frac{H^1(X,\mathbb{C})}{F^1H^1(X)} = H^1(X,\mathcal{O}_X)$, then $J^1(X) = \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}$. By [22, §7.2.2] we have a natural isomorphism $H^{0,1}(X) \cong H^1(X,\mathcal{O}_X)$, it follows that $J^1(X) = \frac{H^{0,1}(X)}{H^1(X,\mathbb{Z})} = T = \mathrm{Pic}^0(X)$, where the last equality holds by Proposition 3.15. □

Proposition 3.31. — *The* 1*-th intermediate Jacobian $J^1(X)$ of a smooth projective variety X is an Abelian variety.*

Proof. — It follows from Proposition 3.30 and Proposition 3.18. □

Remark 3.32. — In general, the k-th intermediate Jacobian $J^{2k-1}(X)$ is a transcendental object even if X is a smooth projective variety whose nature is much more difficult to understand than of $J^1(X)$.

Cohomology group of a curve and its Jacobian. — Recall that a complex smooth projective curve C is an example of complex compact Kähler manifold of dimension 1 (see Example 3.2). In this subsection we will prove an important isomorphism between the first cohomology group of a connected complex smooth projective curve C and the first cohomology group of its Jacobian $J(C)$. More precisely,

Lemma 3.33 **(Fact 3).** — *Let C be a connected complex smooth projective curve. Then the homomorphism $w_* : H^1(J(C),\mathbb{Z}) \to H^1(C,\mathbb{Z})$ on cohomology groups is an isomorphism.*

Proof. — Since in particular C is a complex compact Kähler manifold of dimension 1, by Proposition 3.30 we have that $J(C) = T = \mathrm{Pic}^0(C)$, where $T = \frac{H^{0,1}(C)}{H^1(C,\mathbb{Z})}$ is the complex torus associated to the Hodge structure on $H^1(C,\mathbb{Z})$, so $J(C)$ is itself a Kähler manifold and thus admits a Hodge structure on its group $H^1(J(C))$ (see Remark 3.16), by Lemma 3.17 we have $H^1(J(C),\mathbb{Z}) \cong H^1(C,\mathbb{Z})^*$ and since C is connected and oriented (see [6, C.2.1]) by Poincaré duality (see [22, Theorem 5.30]) we have that $H^1(C,\mathbb{Z})^* \cong H^1(C,\mathbb{Z})$. It follows that $H^1(J(C),\mathbb{Z}) \cong H^1(C,\mathbb{Z})$. □

Remark 3.34. — The existence of the isomorphism in Lemma 3.33 is also true for connected smooth projective curves C over an arbitrary algebraically closed field of characteristic zero, that is, the homomorphism $w_* : H^1_{et}(J(C),\mathbb{Q}_l) \to H^1_{et}(C,\mathbb{Q}_l)$ is an isomorphism (see [1, Remark 4]).

3.4. The Abel–Jacobi Map and The Albanese map. — Let X be a compact Kähler manifold of dimension n.

Definition 3.35 **(The Abel–Jacobi map).** — Let $\mathsf{Z}^k(X)_{\mathrm{hom}}$ be the group of cycles of codimension k homologous to 0 (also called cohomologous to 0), let $J^{2k-1}(X)$ be the k-th intermediate Jacobian. The *Abel–Jacobi map* is a morphism

$$\Phi^k_X : \mathsf{Z}^k(X)_{\mathrm{hom}} \longrightarrow J^{2k-1}(X),$$

defined by $Z \mapsto \Phi^k_X(Z) = \int_\gamma$, where $\gamma \subset X$ is a differentiable chain of dimension $2n-2k+1$ such that $\partial\gamma = Z$ and $\int_\gamma \in \frac{F^{n-k}H^{2n-2k+1}(X)^*}{H_{2n-2k+1}(X,\mathbb{Z})} = J^{2k-1}(X)$, see [22, §12.1.2].

The equality $J^{2k-1}(X) = \frac{F^{n-k}H^{2n-2k+1}(X)^*}{H_{2n-2k+1}(X,\mathbb{Z})}$ in the above definition holds thanks to Poincaré duality ([22, §12.1.2]).
If we want to work in terms of dimension note that for $Z \in \mathsf{Z}_l(X)_{\mathrm{hom}}$ we have the Abel–Jacobi invariant $\Phi_X^{n-l}(Z) \in J^{2(n-l)-1}(X)$.

Lemma 3.36. — *If $Z \in \mathsf{Z}_l(X)_{\mathsf{rat}}$, then $\Phi_X^{n-l}(Z) = 0$ in $J^{2(n-l)-1}(X)$.*

Proof. — See [23, Lemma 9.19]. □

Thanks to the above lemma we prove the existence of the Abel–Jacobi (class) map in the following proposition.

Proposition 3.37. — *There exists a unique homomorphism $\mathsf{CH}_l(X)_{\mathrm{hom}} \to J^{2(n-l)-1}(X)$ from the group of l-cycles on X homologous to 0 modulo rational equivalence to the complex torus $J^{2(n-l)-1}(X)$.*

Proof. — Consider the Abel–Jacobi map $\Phi_X^{n-l} : \mathsf{Z}_l(X)_{\mathrm{hom}} = \mathsf{Z}^{n-l}(X)_{\mathrm{hom}} \to J^{2(n-l)-1}(X)$. Observe that $\mathsf{Z}_l(X)_{\mathsf{rat}}$ is a normal subgroup of $\mathsf{Z}_l(X)_{\mathrm{hom}}$ by Proposition 2.43, so the natural surjective homomorphism $\varphi : \mathsf{Z}_l(X)_{\mathrm{hom}} \to \frac{\mathsf{Z}_l(X)_{\mathrm{hom}}}{\mathsf{Z}_l(X)_{\mathsf{rat}}}$ is well defined.
By Lemma 3.36 we have $\mathsf{Z}_l(X)_{\mathsf{rat}} \subset \mathsf{Ker}(\Phi_X^{n-l} : \mathsf{Z}_l(X)_{\mathrm{hom}} \to J^{2(n-l)-1}(X))$, then by the fundamental theorem on homomorphism, there exists a unique homomorphism $\frac{\mathsf{Z}_l(X)_{\mathrm{hom}}}{\mathsf{Z}_l(X)_{\mathsf{rat}}} = \mathsf{CH}_l(X)_{\mathrm{hom}} \to J^{(2n-2l)-1}(X)$ such that the following diagram commutes

$$\begin{tikzcd} \mathsf{Z}_l(X)_{\mathrm{hom}} \arrow[r, "\Phi_X^{n-l}"] \arrow[dr, "\varphi"] & J^{2(n-l)-1}(X) \\ & \mathsf{CH}_l(X)_{\mathrm{hom}} \arrow[u] \end{tikzcd}$$ □

Remark 3.38. — When $l = 0$, by abuse of notation, the Abel–Jacobi (class) map of Proposition 3.37 is usually denoted by alb_X and called the Albanese map (see [[23], Theorem 10.11]), but there is another map also denoted by alb_X and called the Albanese map which we will define later.

The Abel–Jacobi map for divisors. — Now we give an useful alternative definition of the Abel–Jacobi map Φ_X^k for the case $k = 1$, that is for the case of divisors.
Let $D \in \mathsf{Z}^1(X)$ be a divisor, $[D]$ the cohomology class of D, $\mathcal{O}_X(D)$ the holomorphic line bundle corresponding to the divisor D, and α_D the isomorphism class of $\mathcal{O}_X(D)$.
By Lelong theorem ([22, Theorem 11.33]) $[D] = c_1(\mathcal{O}_X(D))$, then $D \in \mathsf{Z}^1(X)_{\mathrm{hom}}$, i.e., $[D] = 0$ if and only if $c_1(\mathcal{O}_X(D)) = 0$, i.e., $\alpha_D \in \mathrm{Pic}^0(X) = J^1(X)$ (Proposition 3.30). So, α_D is a well defined element in $J^1(X)$.
We also have the following proposition

Proposition 3.39. — $\Phi_X^1(D) = \alpha_D$.

Proof. — See [22, Proposition 12.7]. □

This proposition gives us the following characterization of the Abel- Jacobi map for the case $k = 1$.

***Definition 3.40* (Abel–Jacobi map for divisors).** — Let $\mathsf{Z}^1(X)_{\mathrm{hom}}$ be the group of cycles of codimension 1 homologous to 0 (also called cohomologous to 0 in [22]), let $J^1(X)$ be the 1-th intermediate Jacobian. The *Abel–Jacobi map* is a morphism defined by

$$\begin{array}{rccc} \Phi^1_X: & \mathsf{Z}^1(X)_{\mathrm{hom}} & \longrightarrow & J^1(X) \\ & D & \longmapsto & \Phi^1_X(D)=\alpha_D \end{array}$$

In what follows we prove that the Abel–Jacobi class map for divisors is an isomorphism.

***Definition 3.41* (Alternative definition of rational equivalence).** — Let D be a divisor on X. We say that $D \sim_{\mathsf{rat}} 0$ if it is the divisor of a meromorphic function on X ([22, Definition 12.9]).

Lemma 3.42. — *$\mathcal{O}_X(D)$ is trivial if and only if $D \sim_{\mathsf{rat}} 0$*

Proof. — $\mathcal{O}_X(D)$ is trivial then the meromorphic section σ_D of $\mathcal{O}_X(D)$ whose divisor is equal to D can be seen as a meromorphic function on X thanks to the trivialization, then by Definition 3.41 $D \sim_{\mathsf{rat}} 0$.
Reciprocally, if $D \sim_{\mathsf{rat}} 0$ then by definition D is the divisor of a meromorphic function ϕ on X, then ϕ gives a everywhere non-zero section σ_D (whose divisor is D) of the line bundle $\mathcal{O}_X(D)$, so $\mathcal{O}_X(D)$ is trivial. ☐

Lemma 3.43. — *Let D be a divisor such that $D \sim_{\mathrm{hom}} 0$ on X. Then $\Phi^1_X(D) = 0$ if and only if $D \sim_{\mathsf{rat}} 0$*

Proof. — By Abel's theorem ([22, Corollary 12.8]) we have that $\Phi^1_X(D) = 0$, i.e., $\alpha_D = 0$ if and only if $\mathcal{O}_X(D)$ is trivial. By Lemma 3.42 this last condition is equivalent to $D \sim_{\mathsf{rat}} 0$. ☐

Then we get the following important theorem

Theorem 3.44. — $\mathsf{CH}^1(X)_{\mathrm{hom}} \xrightarrow{\sim} J^1(X)$.

Proof. — From Lemma 3.43 we have $\mathsf{Ker}(\Phi^1_X : \mathsf{Z}^1(X)_{\mathrm{hom}} \to J^1(X)) = \mathsf{Z}^1(X)_{\mathsf{rat}}$. Since Φ^1_X is surjective (see [22, §12.2.2]), and using the first isomorphism theorem of homomorphisms we have that $\mathsf{CH}^1(X)_{\mathrm{hom}} = \frac{\mathsf{Z}^1(X)_{\mathrm{hom}}}{\mathsf{Z}^1(X)_{\mathsf{rat}}} \xrightarrow{\sim} J^1(X)$ is an isomorphism. ☐

Remark 3.45. — Since $J^1(X) = \mathrm{Pic}^0(X)$ we also have that $\mathsf{CH}^1(X)_{\mathrm{hom}} \xrightarrow{\sim} \mathrm{Pic}^0(X)$.

As a easy consequence of the above theorem we get the following corollary which is very important for us.

***Lemma 3.46* (Fact 1).** — *Let C be a smooth projective complex curve, let $J = J(C)$ be the Jacobian of the curve C. Then there exists an isomorphism $alb_C : \mathsf{CH}_0(C)_{\mathrm{deg}=0} \to J$ between the Chow group $\mathsf{CH}_0(C)_{\mathrm{deg}=0}$ of 0-cycles of degree zero on C and the Abelian variety J.*

Proof. — Since C is in particular a compact Kähler manifold (of dimension 1) by Theorem 3.44 we have that the Abel–Jacobi (class) map $alb_C : \mathsf{CH}^1(C)_{\mathrm{hom}} = \mathsf{CH}_0(C)_{\mathrm{hom}} \xrightarrow{\sim} J$ is an isomorphism (see also Remark 3.38), by Fact 2 (see Lemma 2.49) $\mathsf{CH}_0(C)_{\mathrm{hom}} = \mathsf{CH}_0(C)_{\mathrm{deg}=0}$, so we get that $alb_C : \mathsf{CH}_0(C)_{\mathrm{deg}=0} \xrightarrow{\sim} J$ is an isomorphism. Finally, since C is smooth and projective $J = \mathrm{Pic}^0(C)$ is an abelian variety see Proposition 3.18. ☐

Hence we can identify $\mathsf{CH}_0(C)_{\mathrm{deg}=0}$ with J by means of alb_C.

Remark 3.47. — If X is a smooth projective variety of dimension 1 over an arbitrary algebraically closed field there exists a universal pair (A, φ), that is, an abelian variety $A = \mathrm{Pic}^0(X) = J(X) = \mathrm{Alb}(X)$ and a regular homomorphism $\varphi : \mathsf{CH}_0(X)_{\deg=0} = \mathsf{A}_0(X) \to A$ satisfying the universal property (see [17, Notations]). Moreover, the regular homomorphism φ is an isomorphism, see [3] and [1, Remark 2].

The Albanese variety and the Albanese map. — In order to define the Albanese map we need to remember the following theorem due to Griffiths.

Theorem 3.48 **(Griffiths' theorem).** — *Let X be a compact Kähler manifold, Y a connected manifold, $t_0 \in Y$ a reference point, and $Z = \sum n_i Z_i \in \mathsf{Z}^k(Y \times X)$ a cycle of codimension k with each Z_i smooth or flat and such that $\mathsf{pr}_1 : Z_i \to Y$ is a submersion. Then the fibres $Z_t = \sum n_i Z_{i,t}$, where $Z_{i,t} := \mathsf{pr}_1^{-1}(t) \subset X$, are all homologous in X, and the map*

$$\begin{array}{rccc} \phi : & Y & \longrightarrow & J^{2k-1}(X) \\ & t & \longmapsto & \Phi_X^k(Z_t - Z_{t_0}) \end{array}$$

is holomorphic (see [22, Theorem 12.4]*, see also* [22, Remark 12.5]*).*

Griffiths' theorem applied to the following particular case: let X be a connected manifold, $Y = X$, $t_0 = x_0 \in X$, $Z = \Delta \in \mathsf{Z}^n(X \times X)$, where $\Delta = \{(x, y) \in X \times X : x = y\}$ is the diagonal. Give us a holomorphic map

$$\begin{array}{rccc} alb_X : & X & \longrightarrow & J^{2n-1}(X) \\ & x & \longmapsto & \Phi_X^{2n-1}(x - x_0). \end{array}$$

Definition 3.49 **(The Albanese map).** — The map alb_X is called the *Albanese map*

Definition 3.50 **(Albanese variety).** — The complex torus $\mathrm{Alb}(X) := J^{2n-1}(X)$ is called the *Albanese variety* of X.

Example 3.51. — If $\dim(X) = 1$, that is, if X is a curve, then $\mathrm{Alb}(X) = J(X)$.

Property: the image $alb_X(X)$ generates the torus $\mathrm{Alb}(X)$ as a group. More precisely, for sufficiently large r, the morphism

$$\begin{array}{rccc} alb_X^r : & X^r & \longrightarrow & \mathrm{Alb}(X) \\ & (x_1, \ldots, x_r) & \longmapsto & \sum_i alb_X(x_i) \end{array}$$

is surjective. This property implies that if X is a projective variety then $\mathrm{Alb}(X)$ is an Abelian variety ([22, Corollary 12.12]).
There is another important characterization of the Albanese morphism:

Theorem 3.52. — *For any holomorphic map $\psi : X \to T$ from X to a complex torus T such that $\psi(x_0) = 0$, there exists a unique morphism of complex tori $f : \mathrm{Alb}(X) \to T$ such that $\psi = f \circ alb_X$.*

Remark 3.53. — The Abel–Jacobi (class) map $\mathsf{CH}_0(X)_{\mathrm{hom}} \to \mathrm{Alb}(X) = J^{2n-1}(X)$ (Proposition 3.37) is usually also denoted by alb_X and called the Albanese map, see [23, Theorem 10.11].

4. Lefschetz Pencils and The Monodromy Argument

We start this section with the definition of Lefschetz Pencils of hyperplane sections on an n-dimensional smooth projective variety X, then we study the local description of the topology of a Lefschetz degeneration which applied to a Lefschetz pencil $(X_t)_{t\in\mathbb{P}^1}$ of hyperplane sections of X shows that the vanishing cohomology $H^{n-1}(X_t,\mathbb{Z})_{\mathsf{van}}$ is generated by vanishing cycles of a Lefschetz pencil passing through the smooth hyperplane section X_t.
In this section we also study the monodromy action on the cohomology of the fibres of a projective morphism needed to proof item b of Theorem 5.1. We begin with the definition of local systems, then we study the local monodromy for Lefschetz degenerations, which gives us the Picard–Lefschetz formula, next we study the monodromy action associated to the smooth universal hypersurface which gives us the irreducibility of the monodromy action (see Theorem 4.59). The main reference for this section is [23], see also [19].

4.1. Lefschetz Pencils. —

***Definition 4.1* (Pencil of hypersurfaces on a variety).** — Let X be a complex variety, $\mathscr{L}$ a holomorphic line bundle on X and $|\mathscr{L}| := \mathbb{P}(H^0(X,\mathscr{L}))$. A *pencil of hypersurfaces on* X is a projective line $L \cong \mathbb{P}^1$ in $|\mathscr{L}|$.

Remark 4.2. — (Another characterization of a pencil of hypersurfaces) Note that every element $t \in L$ of this pencil is a class of a nonzero and well defined up to a coefficient section σ_t of L. If $X_t \subset X$ denotes the hypersurface on X defined by the section σ_t we get a one to one correspondence between these hypersurfaces X_t and the points $t \in L$, we then write $(X_t)_{t\in L}$ for the *pencil of hypersurfaces of* X.

Assume that $X \subset \mathbb{P}^N$ is a projective subvariety of $\mathbb{P}^N$ and $\mathscr{L} = \mathcal{O}_X(1)$. If the restriction map $H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(X,\mathcal{O}_X(1))$ is an isomorphism, that is, $|\mathscr{L}| := \mathbb{P}(H^0(X,\mathscr{L})) \cong \mathbb{P}(H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(1))) = (\mathbb{P}^N)^*$, where $(\mathbb{P}^N)^*$ is the dual projective space parametrising the hyperplanes of $\mathbb{P}^N$, we have the following definition of a pencil.

***Definition 4.3* (Pencil).** — A *pencil* in $|\mathscr{L}|$ is a line L in $(\mathbb{P}^N)^*$.

***Definition 4.4* (Base locus of a pencil).** — The *base locus* or *axis* of a pencil $(X_t)_{t\in L}$ is defined by $A = \bigcap_{t\in L} X_t \subset X$.

Remark 4.5. — Since $\sigma_t = \sigma_0 + t\sigma_\infty$ for $t \in \mathbb{C} \subset \mathbb{P}^1$, clearly A is defined by the equations: $\sigma_0 = \sigma_\infty = 0$. So $A = \bigcap_{t\in L} X_t = X_0 \cap X_\infty$ is a *complete intersection of codimension* 2 in X if the hypersurfaces X_0 and X_∞ have no common component.

Lefschetz pencil. —

***Definition 4.6* (Lefschetz pencil).** — A pencil $(X_t)_{t\in L}$ of hypersurfaces of X is called a *Lefschetz pencil* if it satisfies the following conditions:

1. The base locus A is smooth of codimension 2 in X. In particular, the hypersurfaces of the pencil are smooth along A.

2. Every hypersurface X_t has at most one ordinary double point as singularity.

In what follows we will give another characterization of Lefschetz pencils.
Let $X \subset \mathbb{P}^N$ be a variety contained in $\mathbb{P}^N$, and assume that X is not degenerate, i.e., the restriction map $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(X, \mathcal{O}_X(1))$ is injective.
For every $t \in (\mathbb{P}^N)^*$ let H_t the hyperplane in $\mathbb{P}^N$ corresponding to t and consider the algebraic subset defined by $Z = \{(x,t) \in X \times (\mathbb{P}^N)^* : X_t := X \cap H_t \text{ is singular at } x\}$. It is known that Z is smooth and $\dim(Z) = N-1$, see [23, §2.1.1].

***Definition 4.7* (The discriminant variety of X/ discriminant locus of $(\mathbb{P}^N)^*$).** — The image $\Delta_X = \mathsf{pr}_2(Z)$ of Z via the second projection is called the *discriminant variety of* X or *discriminant locus of* $(\mathbb{P}^N)^*$.

By definition Δ_X is the set of singular hyperplane sections of X. It is known that $\dim(\Delta_X) \leq N-1$ and that $\dim(\Delta_X) = N-1$ if there exist hyperplane sections of X having an ordinary double point ([23, §2.1.1]).

***Definition 4.8* (A special open subset of the discriminant locus).** — The subset of Δ_X parametrizing hyperplanes H_t such that X_t has at most one ordinary double point as singularity is denoted by Δ_X^0.

Remark 4.9. — If $\dim(\Delta_X) = N-1$ then $\Delta_X^0 \neq \emptyset$ and thus dense, since it is clearly a Zariski open set of Δ_X. Moreover, Δ_X^0 is smooth since pr_2 is an isomorphism over Δ_X^0 ([23, p. 45]).

Then we have the following characterization of Lefschetz pencils

Proposition 4.10. — *Let X be a smooth subvariety of $\mathbb{P}^N$. Then a pencil of hyperplane sections $(X_t)_{t\in L}$ is a Lefschetz pencil if and only if one of the following two conditions is satisfied.*

1. $\dim(\Delta_X) = N-1$, *i.e., the discriminant variety of X is a hypersurface, and the corresponding line $L \subset (\mathbb{P}^N)^*$ to this pencil meets the discriminant hypersurface Δ_X transversely in the open dense set Δ_X^0.*
2. $\dim(\Delta_X) \leq N-2$ *and the corresponding line $L \subset (\mathbb{P}^N)^*$ to this pencil does not meet Δ_X.*

Proof. — See [23, Proposition 2.9]. □

Remark 4.11. — Note that if $p \in \mathbb{P}^N$ is such that $p \in A$ then it lies in every hyperplane of the pencil, and the hyperplanes of the pencil are exactly those containing the axis. Moreover, through any point $p \in \mathbb{P}^N$ such that $p \notin A$ there passes exactly one hyperplane in the pencil, see [15, Chapter 31].

Corollary 4.12. — *If $X \subset \mathbb{P}^N$ is a smooth projective complex variety, then a generic pencil $(X_t)_{t\in L}$ of hyperplane sections of X is a Lefschetz pencil.*

Proof. — See [23, Corollary 2.10]. □

Proposition 4.13. — *If $X \subset \mathbb{P}^N$ is a smooth non-linear surface, then Δ_X is a hypersurface, that is,* $\dim(\Delta_X) = N-1$.

Proof. — See [21, Example 7.5]. □

4.2. Local and Global Lefschetz Theory. —

Local Lefschetz theory. — In this section we study the topology of an ordinary singularity.

***Definition 4.14* (Lefschetz degeneration map).** — Let $B \subset \mathbb{C}^n$ be a ball of radius r centered at $0 \in \mathbb{C}^n$, f the function on B defined by $f(z) = \sum_i z_i^2$, and $B_t := f^{-1}(t)$ the fibre over t. The map f has values in the disk D of radius r^2 and is such that the central fibre B_0 has an ordinary double point at 0 as singularity, whereas the fibres B_t for t near 0 are smooth. The map $f : B \to D$ is called a *Lefschetz degeneration*.

For every point $t = |t|e^{i\theta} \in D^*$ ($D^* = D - \{0\}$) such that $|t| \le r^2$, the fibre B_t contains the sphere S_t^{n-1} defined by

$$S_t^{n-1} = \left\{ z = (z_1, \dots, z_n) \in B : z_i = \sqrt{|t|} e^{i\theta/2} x_i, x_i \in \mathbb{R}, \sum_{1 \le i \le n} x_i^2 = 1 \right\}$$

***Definition 4.15* (Vanishing sphere).** — The sphere S_t^{n-1} contained in the fiber B_t is called a *vanishing sphere of the family* $(B_t)_{t \in D}$.

The name of the sphere S_t^{n-1} is due to the fact that when t tends to 0 (i.e. to the singular point) the sphere tends to contract to a point.

Remark 4.16. — The sphere S_t^{n-1} depends on the choice of coordinates and does not have any privileged orientation. However, its homology class $\delta \in H_{n-1}(B_t, \mathbb{Z})$, defined by the choice of an orientation, is well defined up to sign and is a generator of $H_{n-1}(B_t, \mathbb{Z})$.

***Definition 4.17* (Vanishing cycle).** — The homology class δ of the vanishing sphere S_t^{n-1} is called the *vanishing cycle* of the Lefschetz degeneration $f : B \to D$.

On the other hand, the set $B_{\le |t|} = \{z \in B : |f(z)| \le |t|\}$ contains the ball

$$B_t^n = \left\{ z = (z_1, \dots, z_n) \in B : z_i = \sqrt{|t|} e^{i/2\theta} x_i, x_i \in \mathbb{R}, \sum_{1 \le i \le n} x_i^2 \le 1 \right\}$$

***Definition 4.18* (Cone on the vanishing sphere).** — The ball B_t^n contained in $B_{\le |t|}$ is called the *cone on the vanishing sphere* S_t^{n-1}.

As an application of Morse theory we have

Proposition 4.19. — *For D of small radius with respect to the radius of B, and $s \in D^*$, there exists a retraction by deformation $(H_t')_{t \in [0,1]}$ of $B_D := f^{-1}(D)$ onto the union $B_s \cup B_s^n$ of the fiber B_s with the ball B_s^n. Moreover, this retraction by deformation can be chosen so as to preserve S_D and to be induced on S_D by a retraction $(R_{S,t})_{t \in [0,1]}$ as above.*

Proof. — See [23, Proposition 2.14]. □

A global version of the above proposition states the following. Let $f : X \to D$ be a proper holomorphic map from a n-dimensional complex variety X to a disk such that f a submersion over the punctured disk D^* and that f has a nondegenerate critical point x_0 over $0 \in D$, that is, let f be a Lefschetz degeneration.

Theorem 4.20. — *Then there exists a retraction by deformation of X into the union $X_t \bigcup_{S_t^{n-1}} B_t^n$ of X_t with a n-dimensional ball B_t^n which is glued to X_t along a vanishing sphere $S_t^{n-1} \subset X_t$, where $t \in D^*$.*

Proof. — See [23, Theorem 2.16]. □

Corollary 4.21. — *Let $i : X_t \hookrightarrow X_D$, $t \in D^*$, be the inclusion. Then the homomorphism $i_* : H_k(X_t, \mathbb{Z}) \to H_k(X_D, \mathbb{Z})$ is an isomorphism for $k < n-1$, and is surjective for $k = n-1$. Moreover, the kernel of i_* is generated by the class of "the" vanishing sphere S_t^{n-1} of X_t for $k = n-1$.*

Proof. — See [23, Corollary 2.17]. □

Global theory of Lefschetz. —

***Definition 4.22* (Fibration of topological spaces).** — $\phi : Y \to X$ is called a *fibration of topological spaces* if locally on X there exists a trivialisation of ϕ, i.e., a homeomorphism $Y_U := \phi^{-1}(U) \cong Y_t \times U$ over X, where U is an open neighborhood of $t \in X$.

Example 4.23. — By Ehresmann's Theorem if X and Y are differentiable varieties and ϕ is submersive and proper, then ϕ is a fibration.

Let X be a compact complex variety of dimension n, and let $(X_t)_{t\in\mathbb{P}^1}$ be a Lefschetz pencil on X.
Consider the variety $\widetilde{X} = \{(x,t) \in X \times \mathbb{P}^1 : x \in X_t\}$. Let $\tau = \mathsf{pr}_1|_{\widetilde{X}} : \widetilde{X} \to X$, where $\mathsf{pr}_1 : X \times \mathbb{P}^1 \to X$ is the first projection, and let $f = \mathsf{pr}_2|_{\widetilde{X}} : \widetilde{X} \to \mathbb{P}^1$, where $\mathsf{pr}_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ is the second projection.

- By definition of blowups it is clear that $\widetilde{X} \xrightarrow{\tau} X$ can be identified with the blowup of X along the base locus A of the pencil.
- Each hypersurface X_t of the Lefschetz pencil (hence of X) can be naturally identified with the fibre $f^{-1}(t) \subset \widetilde{X}$ of f.

Since $(X_t)_{t\in\mathbb{P}^1}$ is a Lefschetz pencil, the base locus A is smooth and thus $\widetilde{X}$ is smooth, and since each fibre X_t of f has at most one ordinary double point as singularity we can in the neighborhood of each critical value of f, apply Theorem 4.20 as follows:

- Let $0_i \in \mathbb{P}^1$, with $i = 1, \ldots, M$ be the critical values of f.

  For each i, let D_i be a small disk of $\mathbb{P}^1$ centered at 0_i, and let $\widetilde{X}_{D_i} := f^{-1}(D_i)$. Then $\widetilde{X}_{D_i} \to D_i$ satisfies the property of Theorem 4.20, so we have that $\widetilde{X}_{D_i}$ retracts by deformation onto the union $X_{t_i} \bigcup_{S_{t_i}^{n-1}} B_{t_i}^n$ of X_{t_i} with an n-dimensional ball $B_{t_i}^n$ glued to X_{t_i} along a vanishing sphere $S_{t_i}^{n-1} \subset X_{t_i}$, where $t_i \in D_i^*$.
- Assume that ∞ is not a critical value of f.

  Let $t \in \mathbb{C} = \mathbb{P}^1 - \infty$ be a regular value, and γ_i, $i = 1, \ldots, M$ be the paths in $\mathbb{C}$ joining t to t_i, not passing through the critical values 0_i and meeting only at the point t. Then
  - $\mathbb{C} = \mathbb{P}^1 - \infty$ admits a retraction by deformation onto $\bigcup_{i=1}^M D_i \cup \gamma_i$ the union of the discs D_i with the paths γ_i.

- Since f is a proper fibration above $\mathbb{C}\setminus\{0_1,\ldots,0_M\}$, by Ehresmann's theorem $\widetilde{X}-X_\infty$ admits a retraction by deformation onto $\bigcup_{i=1}^{M}\widetilde{X}_{\gamma_i}\cup\widetilde{X}_{D_i}$, where $\widetilde{X}_{\gamma_i}:=f^{-1}(\gamma_i)$.
- Finally, as f is a fibration above γ_i, each $\widetilde{X}_{\gamma_i}$ admits a trivialization $\widetilde{X}_{\gamma_i}\cong X_{t_i}\times\gamma_i$, above γ_i, and correspondingly, $\widetilde{X}_{\gamma_i}$ admits a retraction by deformation onto X_{t_i}. Moreover this trivialization also gives a diffeomorphism between X_{t_i} and X_t.

So, we have prove the following theorem.

Theorem 4.24. — *(Homotopy type of $\widetilde{X}-X_\infty$) The variety $\widetilde{X}-X_\infty$ has the homotopy type of the union of X_t with n-dimensional balls glued to X_t along $(n-1)$-dimensional spheres.*

Proof. — See [23, Theorem 2.18]. □

Corollary 4.25. — *For $t\in\mathbb{P}^1-\infty$ such that X_t is smooth, the inclusion $i'_t: X_t\hookrightarrow\widetilde{X}-X_\infty$ induces an isomorphism $i'_{t*}: H_k(X_t,\mathbb{Z})\to H_k(\widetilde{X}-X_\infty,\mathbb{Z})$, for $k<n-1$. For $k=n-1$, i'_{t*} is surjective and the kernel of i'_{t*} is generated by the classes of vanishing spheres.*

Proof. — See [23, Corollary 2.20]. □

Remark 4.26. — Note that given a pair (X,Y), where Y is a smooth hyperplane section of $X\subset\mathbb{P}^N$ there exists a Lefschetz pencil $(X_t)_{t\in\mathbb{P}^1}$ of hyperplane sections of X, of which Y is one member X_t ([23, §2.3.2]).

Vanishing cohomology and primitive cohomology. — Let Y be a compact Kähler variety of dimension m, $[\omega]\in H^2(Y,\mathbb{R})$ be a Kähler class. Then we have the operator: $L=[\omega]\cup: H^k(Y,\mathbb{R})\to H^{k+2}(Y,\mathbb{R})$, called the *Lefschetz operator*.

Definition 4.27 **(Primitive cohomology).** — The *primitive cohomology* is defined by

$$H^k(Y,\mathbb{R})_{\mathsf{prim}}:=\mathsf{Ker}(L^{m+1-k}: H^k(Y,\mathbb{R})\longrightarrow H^{2(m+1)-k}(Y,\mathbb{R})).$$

Remark 4.28. — For $k=m$ we have: $H^m(Y,\mathbb{R})_{\mathsf{prim}}:=\mathsf{Ker}(L: H^m(Y,\mathbb{R})\to H^{m+2}(Y,\mathbb{R}))$.

From now on suppose that $Y\overset{j}{\hookrightarrow}X$ is a hyperplane section of a projective variety X of dimension n (hence $m=n-1$). Then we can take $[\omega]=c_1(\mathcal{O}_Y(1))=h_Y$, and the equality: $j^*\circ j_*=h_Y\cup: H^k(Y,\mathbb{Z})\to H^{k+2}(Y,\mathbb{Z})$ says that the corresponding Lefschetz operator of $[\omega]$ satisfies: $L=j^*\circ j_*: H^k(Y,\mathbb{R})\xrightarrow{j_*}H^{k+2}(X,\mathbb{R})\xrightarrow{j^*}H^{k+2}(Y,\mathbb{R})$.

Definition 4.29 **(Vanishing cohomology).** — For every coefficient ring R, the *vanishing cohomology* is defined by

$$H^k(Y,R)_{\mathsf{van}}=\mathsf{Ker}(j_*: H^k(Y,R)\longrightarrow H^{k+2}(X,R)).$$

For the case $k=n-1=\dim(Y)$, the following is an important property of the vanishing cohomology

Lemma 4.30. — *The vanishing cohomology $H^{n-1}(Y,\mathbb{Z})_{\mathsf{van}}$ is generated by the classes of vanishing spheres of a Lefschetz pencil passing through Y.*

Proof. — See [23, Lemma 2.26]. □

Lemma 4.31. — *The vanishing cohomology $H^{n-1}(Y,\mathbb{Z})_{\mathsf{van}}$ is a Hodge substructure.*

Proof. — By Proposition 3.26 j_* is a morphism of Hodge structures and by Remark 3.22 $\ker(j_*) = H^{n-1}(Y,\mathbb{Z})_{\mathsf{van}}$ has the structure of Hodge structure. □

***Proposition 4.32* (Comparison between primitive and vanishing cohomology).** —

1. *There is a decomposition as an orthogonal direct sum (relative to the intersection form in $H^{n-1}(Y,\mathbb{Q})$): $H^{n-1}(Y,\mathbb{Q}) = H^{n-1}(Y,\mathbb{Q})_{\mathsf{van}} \oplus j^*H^{n-1}(X,\mathbb{Q})$.*

2. *Similarly, there is a decomposition as an orthogonal direct sum: $H^{n-1}(Y,\mathbb{Q})_{\mathsf{prim}} = H^{n-1}(Y,\mathbb{Q})_{\mathsf{van}} \oplus j^*H^{n-1}(X,\mathbb{Q})_{\mathsf{prim}}$.*

Proof. — See [23, Lemma 2.27]. □

4.3. Monodromy of Lefschetz Pencils. —

Local systems on topological spaces. — Let X be a topological space and let A be some commutative ring with a unit.

***Definition 4.33* (Local system of A-modules on X).** — A *local system of A-modules on X* consists of the following data: for each $t \in X$ an A-module G_t and for any two points $t, t' \in X$ a collection of isomorphisms $\rho([\gamma]) : G_t \xrightarrow{\sim} G_{t'}$, one for each homotopy class $[\gamma]$ of paths from t to t'. A local system with fibres G_t is usually denoted by $\mathcal{G}$.

***Definition 4.34* (Constant local system).** — The *constant local system with fibre G* is denoted by G_X.

***Definition 4.35* (Monodromy representation associated to a local system).** — Let (X,t) be a pointed path connected (i.e., 0-connected) topological space, the collection $\{\rho([\gamma]) : G_t \to G_t | \gamma \text{ a loop at } t\}$ defines the associated *monodromy representation*

$$\begin{array}{rccc} \rho: & \pi_1(X,t) & \longrightarrow & \mathrm{GL}(G_t) \\ & [\gamma] & \longmapsto & \rho([\gamma]) \end{array}$$

***Remark 4.36* (In a locally simply connected space local systems are locally constant local systems).** — Let $\mathcal{G}$ a local system on the topological space X. If X is a locally simply connected space, i.e., locally 1-connected, then it admits a covering $\{U_i\}_{i\in I}$ by simply connected open subsets therefore for any two points $t, t' \in U_i$ there is a unique homotopy class $[\gamma]$ of paths from t to t' inside U_i, so there is a unique isomorphism $f_{t,t'} : G_t \xrightarrow{\sim} G_{t'}$ defined by any path connecting t and t' in U_i. This gives a canonical trivialization of the local system $\mathcal{G}$ above U_i, say $\phi_i : \mathcal{G}|U_i \xrightarrow{\sim} G_{U_i}$.

Let X be a path connected and locally simply connected space. Then we have the following property

Lemma 4.37. — *Let X be a path connected and locally simply connected space. Then there is a one to one correspondence between locally constant sheaves of A-modules and local systems of A-modules on X.*

Proof. — See [19, Lemma B.34.]. By Remark 4.36 note that the one to one correspondence is actually with locally constant local systems of A-modules on X. □

In a locally connected topological space we have the following alternative definition of a local systems, see [23, §3.1.1].

***Definition 4.38* (Local system of stalk G).** — Let X be a locally connected topological space, and let G be an abelian group. A *local system of stalk* G is a sheaf which is locally isomorphic to the constant sheaf of stalk G.

For another definition of constant sheaf see [22, Example 4.5].

***Definition 4.39* (Local system of A-modules of stalk G).** — Let G be an A-module. A *local system of A-modules of stalk* G is a sheaf of A-modules which is locally isomorphic, as a sheaf of A-modules, to the constant sheaf of stalk G.

The following corollary gives a relation of local systems and representations.

Corollary 4.40. — *If X is arcwise connected and locally simply connected and $x \in X$, we have a natural bijection between the set of isomorphism classes of local systems of stalk G, and the set of representations $\pi_1(X, x) \to \mathsf{Aut}(G)$, modulo the action of $\mathsf{Aut}(G)$ by conjugation.*

Proof. — See [23, Proposition 3.0]. ☐

***Definition 4.41* (Monodromy representation).** — The representation (of $\pi_1(X, x)$)

$$\rho : \pi_1(X, x) \longrightarrow \mathsf{Aut}(\mathcal{G}_x) = \mathsf{Aut}(G)$$

corresponding to a local system is called the *monodromy representation*.

Now we study local systems associated to a fibration.
Let $\phi : Y \to X$ be a fibration of topological spaces (see Definition 4.22) and assume that X is locally contractible. Then for sufficiently small U, the open sets $Y_U = \phi^{-1}(U)$ have the same homotopy type as the fibre $Y_u = \phi^{-1}(u)$ with $u \in U$. Therefore, using the invariance under homotopy of U, one deduces that for every ring of coefficients A, the sheaves $R^k\phi_*A$ are locally constant sheaves. Recall that $R^k\phi_*$ is the right k-th derived functor of the functor $\phi_* :$ Category of sheaves on $Y \to$ Category of sheaves on X.

Proposition 4.42. — *The monodromy representation $\rho : \pi_1(X, x) \to \mathsf{Aut}(H^k(Y_x, A))$ of $\pi_1(X, x)$ on the stalk $H^k(Y_x, A) = (R^k\phi_*A)_x$ of the local system $R^k\phi_*A$ is induced by homeomorphisms of the fibre Y_x.*

Proof. — See [23, p. 74]. ☐

In what follows we study some restrictions that Hodge theory imposes on the monodromy representation.

***Definition 4.43* (Projective morphism).** — A morphism $\phi : Y \to X$ of complex varieties is called *projective* if there exists a holomorphic immersion $i : Y \hookrightarrow X \times \mathbb{P}^N$ such that $\mathsf{pr}_1 \circ i = \phi$.

Let $\phi : Y \to X$ be a holomorphic, submersive and projective morphism of complex varieties. Then we have a monodromy representation $\rho : \pi_1(X, x) \to \mathsf{Aut}(H^k(Y_x, \mathbb{Z}))$, as above for every k.
By Hogde theory we know that every group $H^k(Y_x, \mathbb{Z})$ is equipped with a Hodge structure (see Example 3.5).
The first restriction imposed by Hodge theory on ρ is

Proposition 4.44. — *$H^k(Y_x,\mathbb{Z})^\rho := \{\alpha \in H^k(Y_x,\mathbb{Z}) \mid \rho(\gamma)(\alpha) = \alpha,\ \forall\ \gamma \in \pi_1(X,x)\}$, the space of invariants under ρ, is a Hodge substructure $H^k(Y_x,\mathbb{Z})$ if X is quasi-projective.*

Proof. — See [23, Proposition 3.14]. □

Another consequence of Hodge theory concerns the local monodromy called the quasi-unipotence theorem.

Theorem 4.45 **(Quasi-unipotence theorem).** — *Let X be a punctured disk D^*, so that $\pi_1(X,x) = \mathbb{Z}$ and the monodromy group $\mathrm{im}(\rho) \subset \mathsf{Aut}(H^k(Y_x,\mathbb{Z}))$ is generated by a single element T. Then T is quasi-unipotent, i.e., there exists integers N and M such that $(T^N - 1)^M = 0$. In fact, we can even take $M \leq k+1$.*

Proof. — See [23, Proposition 3.15]. □

The Picard–Lefschetz formula for a Lefschetz degeneration. — In a wider context we have the following definition of a Lefschetz degeneration (Definition 4.14).

Definition 4.46 **(Lefschetz degeneration).** — Let X be a smooth n-dimensional complex variety. The map $f : X \to D$ is called a *Lefschetz degeneration* if f is proper with non-zero differential over the punctured disc D^*, and such that the fibre X_0 has an ordinary double point as its unique singularity.

Let X be a smooth n-dimensional complex variety and $f : X \to D$ a Lefschetz degeneration. Let $t \in D^*$. Since in this case $\pi_1(D^*,t) = \mathbb{Z}$, the monodromy representation $\rho : \pi_1(D^*,t) \to \mathsf{Aut}(H^{n-1}(X_t,\mathbb{Z}))$ on the cohomology of the fibre $H^{n-1}(X_t,\mathbb{Z})$ is determined by $T \in \mathsf{Aut}(H^{n-1}(X_t,\mathbb{Z}))$, where T denotes the image via ρ of the generator of $\pi_1(D^*,t)$.
Let $\delta \in H^{n-1}(X_t,\mathbb{Z})$ be the cohomology class of the sphere $S_t^{n-1} \subset X_t$ defined by an orientation, and recall that δ is a generator of $\mathsf{Ker}(H^{n-1}(X_t,\mathbb{Z}) \cong H_{n-1}(X_t,\mathbb{Z}) \to H_{n-1}(X,\mathbb{Z}))$, (see Corollary 4.21).
Recall also that the fiber X_t is a real oriented $(2n-2)$-dimensional variety, so we have the intersection form $\langle \cdot , \cdot \rangle$ on $H^{n-1}(X_t,\mathbb{Z})$.
Then we have the following important theorem

Theorem 4.47 **(Picard–Lefschetz Theorem).** — *For every $\alpha \in H^{n-1}(X_t,\mathbb{Z})$ we have: $T(\alpha) = \alpha + \epsilon_n\langle\alpha,\delta\rangle\delta$, where $\epsilon_n = \pm 1$ according to the value of n.*

Proof. — See [23, Theorem 3.16]. □

Monodromy action associated to the smooth universal hypersurface and Zariski's theorem. —

Definition 4.48 **(Family of projective varieties $\mathbb{P}^N$).** — Let T be a variety. A *family of projective varieties in the projective space $\mathbb{P}^N$ with base T* is a closed subvariety $\mathscr{X}$ of the product $\mathbb{P}^N \times T$. The fibers $X_t = p_2^{-1}(t)$ over points $t \in T$ are called the *members* or *elements* of the family; the variety $\mathscr{X}$ is called the *total space* and the family is said to be *parametrized* by T.

Example 4.49 **(The universal family).** — For any closed point $t \in \mathbb{P}^{N*}$ let H_t be the corresponding hyperplane in $\mathbb{P}^N$. The subset of $\mathbb{P}^N \times \mathbb{P}^{N*}$ defined by

$$\mathscr{H} = \left\{(x,t) \in \mathbb{P}^N \times \mathbb{P}^{N*} : x \in H_t\right\}$$

is a subvariety of $\mathbb{P}^N \times \mathbb{P}^{N*}$. Since the fibers over $\mathbb{P}^{N*}$, via $p_2 : \mathscr{H} \to \mathbb{P}^{N*}$, are all hyperplanes in $\mathbb{P}^N$ we think of $\mathscr{H}$ as the family of hyperplanes in $\mathbb{P}^N$ parametrized by $\mathbb{P}^{N*}$. Needless to say that the situation is symmetric, so we may also view $\mathscr{H}$, via $p_1 : \mathscr{H} \to \mathbb{P}^N$, as the family of all hyperplanes in $\mathbb{P}^{N*}$ parameterized by $\mathbb{P}^N$. $\mathscr{H}$ is called *the universal family* (see [9, Lecture 4]).

Remark 4.50. — The adjective *universal* of $\mathscr{H}$ is due to the fact that if $\mathscr{X}_T \subset \mathbb{P}^N \times T$ is any flat family of hyperplanes (parametrized by T) then there is a unique regular map $T \to \mathbb{P}^{N*}$ such that $\mathscr{X}_T$ is the fiber product $T \times_{\mathbb{P}^{N*}} \mathscr{H}$.

***Example 4.51* (Universal hyperplane section).** — Let $X \subset \mathbb{P}^N$ be a projective variety, $\mathscr{H}$ the universal hyperplane, and $p_1 : \mathscr{H} \to \mathbb{P}^d$ is the projection on the first factor. Set

$$\mathscr{C} = \{(x,t) \in \mathscr{H} : x \in H_t \cap X\} = p_1^{-1}(X)$$

By the second description $\mathscr{C}$ is a subvariety of $X \times \mathbb{P}^{d*}$. Let f be the composition of the closed embedding $\mathscr{C} \hookrightarrow \mathscr{H}$ and $p_2 : \mathscr{H} \to \mathbb{P}^{d*}$. Since the fibers over $\mathbb{P}^{N*}$, via f, are all hyperplane sections of X we think of $\mathscr{C}$ as the family of hyperplane sections of X parametrized by $\mathbb{P}^{N*}$. This family is called the *universal hyperplane section* of X, see [9, Lecture 4].

Let $X \subset \mathbb{P}^N$ be a smooth projective connected non-degenerate variety of dimension n. Let $\Delta_X = \mathsf{pr}_2(Z) \subset (\mathbb{P}^N)^*$ be the discriminant variety of X, i.e., the set of singular hyperplane sections of X. An important property of Δ_X is that it is irreducible since it is the image in $(\mathbb{P}^N)^*$ of the smooth irreducible variety Z (see Definition 4.7). Let $U := (\mathbb{P}^N)^* \smallsetminus \Delta_X$ be complement of Δ_X.

***Definition 4.52* (Smooth universal hyperplane section).** — Set

$$\mathscr{C}_U = \{(x,t) \in X \times U : x \in X_t = X \cap H_t\}$$

and $f_U : \mathscr{C}_U \to U$. Since the fibers over U are smooth hyperplane sections of X we think of $\mathscr{C}_U$ as the family of smooth hyperplane sections of X parametrized by U. This family is called the *smooth universal hyperplane section* of X. Note that by definition of U, f_U is a submersion.

Now in the following remark we recall an important fact about Lefschetz pencils of hyperplane sections of X.

Remark 4.53. — Let $L \subset \mathbb{P}^{N*}$ be a Lefschetz pencil through $t \in U$, and recall that Δ_X^0 is the open dense subset of Δ_X parametrizing hyperplane sections X_t having exactly one ordinary double point (see also [13, §1.5]).

- Case 1. If $\dim(\Delta_X) \leq N-2$, then the Lefschetz pencil L does not meet Δ_X at all (see Proposition 4.10). In this case $\pi_1(U,t) = 1$, so there is no monodromy action associated to the fibration f_U, see [23, §3.2.2].
- Case 2. If $\dim(\Delta_X) = N-1$, then the Lefschetz pencil L meets Δ_X transversely in its smooth locus Δ_X^0 (see Proposition 4.10).

In the second case, Zariski's theorem shows that the monodromy representation $\rho : \pi_1(U,t) \to \mathsf{Aut}(H^k(X_t, \mathbb{Z}))$ can be computed by restricting to a Lefschetz pencil.

***Theorem 4.54* (Zariski's theorem).** — *Let $\mathcal{Y} \subset \mathbb{P}^r$ be a hypersurface, and let $U = \mathbb{P}^r \smallsetminus \mathcal{Y}$ be its complement. Then for $t \in U$ and for every projective line $L \subset \mathbb{P}^r$ passing through t which meets $\mathcal{Y}$ transversally in its smooth locus, the map $\pi_1(L - L \cap \mathcal{Y}, t) \to \pi_1(U, t)$ is surjective.*

Proof. — See [23, Theorem 3.22]. □

Next we prove that if we are in the second case of the above remark, the vanishing cycles are conjugate under the monodromy action.
Assume that $\dim(\Delta_X) = N - 1$, i.e., Δ_X is a hypersurface. Fix any $t \in U$, then we have the monodromy representation $\rho : \pi_1(U, t) \to \mathsf{Aut}(H^{n-1}(X_t, \mathbb{Z}))$, associated to the fibration f_U.
Moreover, for every $y \in \Delta_X^0$, let $y' \in U$ be near y, contained in a disk D_y which meets Δ_X^0 transversally at y, and such that $D_y \smallsetminus \{y\} \subset U$. Then we have a vanishing cycle (of the Lefschetz degeneration $X_{D_y} \to D_y$ obtained by restricting f_U to $X_{D_y} = f_U^{-1}(D_y)$)

$$\delta_y \in H^{n-1}(X_{y'}, \mathbb{Z}) = H_{n-1}(X_{y'}, \mathbb{Z}), \text{ where } X_{y'} := f_U^{-1}(y')$$

i.e., the homology class of the sphere $S_{y'}^{n-1} \subset X_{y'}$ which is well defined up to sign as a generator of the kernel of the map $H_{n-1}(X_{y'}, \mathbb{Z}) \to H_{n-1}(X_{D_y}, \mathbb{Z})$, see Corollary 4.25.
Now choose a path γ from t to y' contained in U; then, by trivialising the fibration f_U over γ, we can construct a diffeomorphism $\psi : X_{y'} \cong X_t$, well-defined up to homotopy. Thus, we have a vanishing cycle $\delta_\gamma = \psi_*(\delta_y) \in H_{n-1}(X_t, \mathbb{Z}) = H^{n-1}(X_t, \mathbb{Z})$, where $\psi_* : H_{n-1}(X_{y'}, \mathbb{Z}) \to H_{n-1}(X_t, \mathbb{Z})$.
Then thanks to the fact that Δ_X is irreducible we obtain the following result

Proposition 4.55. — *All the vanishing cycles δ_γ (one for each $y \in \Delta_X^0$) constructed above (and defined up to sign) are conjugate (up to sign) under the monodromy action ρ.*

Proof. — See [23, Proposition 3.23]. □

Corollary 4.56. — *Let $(X_t)_{t \in \mathbb{P}^1}$ be a Lefschetz pencil of hyperplane sections of X, 0_i, $i = 1, \ldots, M$ the critical values, and $t \in \mathbb{P}^1$ a regular value. Then all the vanishing cycles $\delta_i \in H^{n-1}(X_t, \mathbb{Z})$ of the pencil are conjugate under the monodromy action of $\rho : \pi_1(\mathbb{P}^1 \smallsetminus \{0_1, \ldots, 0_M\}, t) \to \mathsf{Aut}(H^{n-1}(X_t, \mathbb{Z}))$.*

Proof. — It follows from Theorem 4.54 and the proposition above. □

The following proposition tell us the vanishing cohomology of a smooth hyperplane section is stable under the monodromy action associated to f_U.

Proposition 4.57. — *Let X_t be a smooth hyperplane section of X. Then the monodromy action $\rho : \pi_1(U, t) \to \mathsf{Aut}(H^{n-1}(X_t, \mathbb{Q}))$, associated to the fibration f_U, leaves $H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}}$ stable.*

Proof. — See [23, p. 88]. □

***Definition 4.58* (Irreducible action).** — The action of a group G on a vector space E is said to be *irreducible* if every vector subspace $F \subset E$ stable under G is equal to $\{0\}$ or E.

The following theorem gives the irreducibility of the monodromy representation for smooth hyperplane sections X_t of X, that is, that there is no non-trivial local subsystem of the local system with stalk $H^{n-1}(X_t, \mathbb{Q})_{van}$.

Theorem 4.59. — *Let $(X_t)_{t\in\mathbb{P}^1}$ be a Lefschetz pencil of hyperplane sections of X, 0_i, $i = 1,\dots,M$ the critical values, and $t \in \mathbb{P}^1$ a regular value. Then the monodromy action $\rho : \pi_1(U,t) \to \mathsf{Aut}(H^{n-1}(X_t,\mathbb{Q})_{van})$ is irreducible.*

Proof. — See [23, Theorem 3.27]. □

5. The Gysin Kernel

In this section we present and prove the main result of the paper called the *theorem on the Gysin kernel*.

5.1. A theorem on the Gysin kernel. — Let S be a connected smooth projective surface over $\mathbb{C}$, D a very ample divisor on S and $\mathcal{O}_S(D)$ its corresponding very ample invertible sheaf on S. Let $\Sigma = |D| = |\mathcal{O}_S(D)|$ be the complete linear system of D on S, $d = \dim(\Sigma)$, and $\phi_\Sigma : S \hookrightarrow \mathbb{P}^d$ the closed embedding of S on $\mathbb{P}^d$, induced by Σ (see [11, II, Section 7]).
Let $\mathbb{P}^{d*}$ be the dual projective space of $\mathbb{P}^d$ parametrizing hyperplanes in $\mathbb{P}^d$ and let $\overline{\eta}$ be its geometric generic point. Recall that by definition the linear system Σ can be identified with $\mathbb{P}^{d*}$.
For any closed point $t \in \Sigma = \mathbb{P}^{d*}$, let H_t be the corresponding hyperplane in $\mathbb{P}^d$, $C_t = H_t \cap S$ the corresponding hyperplane section of S, and $r_t : C_t \hookrightarrow S$ the closed embedding of the curve C_t into S.
Let $\Delta_S = \{t \in \Sigma : C_t \text{ is singular}\}$ be the subset in Σ parametrizing singular hyperplane sections of S and called the discriminant variety of S also called the discriminant locus of Σ (see Definition 4.7).
Let $U = \Sigma \setminus \Delta_S = \{t \in \Sigma : C_t \text{ is smooth}\}$ be the complement of Δ_S parametrizing smooth hyperplane sections of S.
For any closed point $t \in U$, let $r_{t*} : H^1(C_t,\mathbb{Z}) \to H^3(S,\mathbb{Z})$ be the Gysin homomorphism on cohomology groups induced by r_t (see Definition 3.25). Recall that $H^1(C_t,\mathbb{Z})$ and $H^3(S,\mathbb{Z})$ carries a weight 1 and 3 Hodge structure respectively (see Example 3.5) and that r_{t*} is a morphism of Hodge structures of bidegree $(1,1)$ (see Proposition 3.26). Let $H^1(C_t,\mathbb{Z})_{\mathsf{van}}$ be the kernel of the above Gysin homomorphism r_{t*}, it is called the *vanishing cohomology* (see Definition 4.29) and it carries a Hodge structure induced by the morphism of Hodge structures r_{t*} (see Lemma 4.31).
For any closed point $t \in U$, let $J_t = J(C_t)$ be the Jacobian of the curve C_t. Recall that J_t is the complex torus associated to the Hodge structure of weight 1 on $H^1(C_t,\mathbb{Z})$ (see Proposition 3.30) and since C_t is smooth and projective it is an Abelian variety (see Proposition 3.31). Let B_t be the abelian subvariety of the abelian variety J_t corresponding to the Hodge substructure on $H^1(C_t,\mathbb{Z})_{\mathsf{van}}$.
For any closed point $t \in U$, let $r_{t*} : \mathsf{CH}_0(C_t)_{\deg=0} \to \mathsf{CH}_0(S)_{\deg=0}$ be the Gysin homomorphism on the Chow groups of 0-cycles of degree zero of C_t and S respectively (see Definition 2.15), induced by r_t (see Definition 2.11), whose kernel $G_t = \mathsf{Ker}(r_{t*})$ will be called the *Gysin kernel* associated to the hyperplane section C_t.

***Theorem 5.1* (A theorem on the Gysin kernel).** — *Let S, Δ_S, U, $\overline{\eta}$, G_t, B_t and J_t be as above. Then*

a. *For every $t \in U$, there is an abelian subvariety A_t of $B_t \subset J_t$ such that*

$$G_t = \bigcup_{\text{countable}} \textit{translates of } A_t$$

b. *For very general $t \in U$ either, $A_t = 0$ or $A_t = B_t$.*

For the above we mean:

There exits a c-open subset $U_0 \subset U$ such that either $A_{\overline{\eta}} = 0$, in which case $A_t = 0$ for all $t \in U_0$, or $A_{\overline{\eta}} = B_{\overline{\eta}}$, in which case $A_t = B_t$ for all $t \in U_0$.

Remark 5.2. — In item b note that if $A_t = 0$ then it follows immediately by item a that G_t is countable and if $A_t = B_t$ and then it follows immediately by item a that G_t is a countable union of translates of B_t.

Proof of item a of Theorem 5.1. — Recall that $U = \Sigma \setminus \Delta_S$ is the open subset of $\Sigma = \mathbb{P}^{d*}$ parametrizing smooth hyperplane sections of S.
Let $t \in U = \Sigma \setminus \Delta$ be any (closed) point in U, so the corresponding section C_t is a smooth (hence connected, see [8]) curve of S and $r_t : C_t \hookrightarrow S$ is the closed embedding of the smooth connected curve C_t into S.
For each natural number d, let $\mathsf{Sym}^d(C_t)$ be the d-th symmetric product of the curve C_t, $\mathsf{Sym}^d(S)$ the d-th symmetric product of the surface S (see Definition 2.21), $\mathsf{Sym}^d(r_t) : \mathsf{Sym}^d(C_t) \to \mathsf{Sym}^d(S)$ the morphism from the d-th symmetric product of the curve C_t to the d-th symmetric product of the surface S, induced by the closed embedding $r_t : C_t \hookrightarrow S$, and

$$\mathsf{Sym}^{d,d}(r_t) : \mathsf{Sym}^{d,d}(C_t) = \mathsf{Sym}^d(C_t) \times \mathsf{Sym}^d(C_t) \longrightarrow \mathsf{Sym}^{d,d}(S) = \mathsf{Sym}^d(S) \times \mathsf{Sym}^d(S).$$

We obtain the following commutative diagram

$$\begin{array}{ccc} \mathsf{Sym}^{d,d}(C_t) & \xrightarrow{\mathsf{Sym}^{d,d}(r_t)} & \mathsf{Sym}^{d,d}(S) \\ \Big\downarrow{\scriptstyle\theta_d^{C_t}} & & \Big\downarrow{\scriptstyle\theta_d^{S}} \\ \mathsf{CH}_0(C_t)_{\deg=0} & \xrightarrow{r_{t*}} & \mathsf{CH}_0(S)_{\deg=0} \end{array} \tag{1}$$

where $\theta_d^{C_t}$ and θ_d^S are the set-theoretic maps of Definition 2.22 (see also Remark 2.23), r_{t*} is the Gysin homomorphism on Chow groups of 0-cycles of degree 0 induced by r_t.
Now recall that by Lemma 3.46 (Fact 1) and Lemma 2.49 (Fact 2) there exists an isomorphism $alb_{C_t} : \mathsf{CH}_0(C_t)_{\deg=0} \to J_t =: \mathrm{Alb}(C_t)$ called the Albanese map. Here $\mathrm{Alb}(C_t)$ is the Albanese variety of C_t. Then, by Definition 2.31, $\mathsf{CH}_0(C_t)_{\deg=0}$ is representable, or equivalently $\theta_d^{C_t} : \mathsf{Sym}^{d,d}(C_t) \to \mathsf{CH}_0(C_t)_{\deg=0}$ is surjective for sufficiently large d, by Definition 2.30. This implies that the Gysin kernel is of the form:

$$G_t = \theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)].$$

Indeed, by the commutativity of the diagram (1) we have

$$(r_{t*} \circ \theta_d^{C_t})^{-1}(0) = (\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0),$$

then $\theta_d^{C_t}[(r_{t*} \circ \theta_d^{C_t})^{-1}(0)] = \theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)]$, then by properties of the inverse of a composition and by the surjectivity of $\theta_d^{C_t}$ we get $r_{t*}^{-1}(0) = \theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)]$, i.e., $G_t = \theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)]$.

On the other hand, by Lemma 2.32 $(\theta_d^S)^{-1}(0)$ is a countable union of Zariski closed subsets in $\mathsf{Sym}^{d,d}(S)$. It follows that $(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)$ is the countable union of Zariski closed subsets in $\mathsf{Sym}^{d,d}(C_t)$.

Now for each d, consider the composition $\mathsf{Sym}^{d,d}(C_t) \overset{\theta_d^{C_t}}{\to} \mathsf{CH}_0(C_t)_{\deg=0} \overset{alb_{C_t}}{\to} J_t$, by Lemma 2.27 it follows that the set-theoretic map $\theta_d^{C_t}$ is regular and by Lemma 2.29 the composition $alb_{C_t} \circ \theta_d^{C_t}$ is a morphism of varieties. Since these varieties are projective, this composition is proper (so it takes closed subsets to closed subsets). It follows that $alb_{C_t}(G_t) = alb_{C_t}(\theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)]) = alb_{C_t} \circ \theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)]$ is also a countable union of Zariski closed subsets in the abelian variety J_t.
Now since $alb_{C_t}(G_t)$ is a countable union of algebraic varieties over $\mathbb{C}$ (which is uncountable), $alb_{C_t}(G_t)$ admits a unique irredundant decomposition inside the abelian variety J_t, see Lemma 2.18. Using the isomorphism alb_{C_t} we can identify $alb_{C_t}(G_t)$ with G_t and write $G_t = \bigcup_{n\in\mathbb{N}}(G_t)_n$ for the irredundant decomposition of G_t in $J_t \simeq \mathsf{CH}_0(C_t)_{\deg=0}$.
On the other hand, note that by definition G_t is a subgroup in $\mathsf{CH}_0(C_t)_{\deg=0}$, hence its image $alb_{C_t}(G_t)$ in J_t via alb_{C_t} is also a subgroup in J_t.
As J_t is an abelian variety and $G_t \subset J_t$ a subgroup which can be represented as a countable union of Zariski closed subsets in J_t, the irredundant decomposition of G_t contains a unique irreducible component passing through 0 which is an abelian subvariety of J_t (see Lemma 2.19). After renumbering of the components, we may assume that this component is $(G_t)_0$.
It is clear that for any $x \in G_t$, the set $x + (G_t)_0$ is an irreducible Zariski closed subset (just translation of a Zariski closed subset) in G_t, and that we can write $G_t = \bigcup_{x\in G_t}(x + (G_t)_0)$.
Ignoring each set $x + (G_t)_0$ inside $y + (G_t)_0$, for $x, y \in G_t$, we get a subset $\Xi_t \subset G_t$ such that $G_t = \bigcup_{x\in\Xi_t}(x + (G_t)_0)$ which is an irredundant decomposition of G_t.
Now we claim that Ξ_t is countable. Indeed, for any $x, y \in \Xi_t$, $x + (G_t)_0$ and $y + (G_t)_0$ are irreducible and not contained one in another. Now observe that since $x+(G_t)_0$ is irreducible and G_t is a subgroup, then $x+(G_t)_0 \subset (G_t)_n$ for some n, because otherwise if $x+(G_t)_0$ is not contained in $(G_t)_n$ for every $n \in \mathbb{N}$, then $x + (G_t)_0$ would be the union of the closed subsets of the form $(x + (G_t)_0) \cap (G_t)_n$ each of which is not $x + (G_t)_0$, contradicting Lemma 2.16. It follows that $(G_t)_0 \subset -x + (G_t)_n$. Similarly, we can prove that $-x + (G_t)_n$ is contained in $(G_t)_l$ for some $l \in \mathbb{N}$. Then $(G_t)_0 = (G_t)_l$, that is, $(G_t)_0 \subset -x + (G_t)_n \subset (G_t)_0$, so $-x + (G_t)_n = (G_t)_0$, i.e., $x + (G_t)_0 = (G_t)_n$ for each $x \in \Xi_t$. It means that Ξ_t is countable.
Taking $A_t = (G_t)_0$, until now we have proved that there is there is an Abelian variety $A_t \subset J_t$ such that $G_t = \bigcup_{x\in\Xi_t}(x + A_t)$, where $\Xi_t \subset G_t$ is a countable subset. Equivalently, we can write as follows: there is an Abelian variety $A_t \subset J_t$ such that $G_t = \bigcup_{\text{countable}}$ translates of A_t.
To complete the proof of item (a) we next show that A_t is contained in B_t.
Let $i : A_t \hookrightarrow J_t$ be the closed embedding of A_t into J_t. Fix an ample line bundle $\mathcal{L}_t$ on J_t, and let $\mathcal{L}'_t$ be the pullback of $\mathcal{L}_t$ to A_t under the embedding i. Then we have an homomorphism $\lambda_{\mathcal{L}'_t} : A_t \to A_t^\vee$ from the abelian subvariety A_t to its dual induced by $\mathcal{L}'_t$, see [14, Chapter 8]. By Remark 8.7 in [14] we have that $\dim(A_t) = \dim(A_t^\vee)$. Then we have the Gysin homomorphism $(\lambda_{\mathcal{L}'_t})_* : H^1(A_t, \mathbb{Z}) \to H^1(A_t^\vee, \mathbb{Z})$ on cohomology groups induced by $\lambda_{\mathcal{L}'_t}$ (see Definition 3.25).
Let $i^\vee : J_t^\vee \hookrightarrow A_t^\vee$ be the homomorphism on dual abelian varieties (see [14, Chapter 9]) induced by the closed embedding $i : A_t \hookrightarrow J_t$, this induces the pullback homomorphism $(i^\vee)^* : H^1(A_t^\vee, \mathbb{Z}) \to H^1(J_t^\vee, \mathbb{Z})$ on cohomology groups associated to $i^\vee$.

Let $\lambda_{\mathcal{L}_t} : J_t \to J_t^\vee$ be the homomorphism from the abelian variety J_t to its dual induced by $\mathcal{L}_t$, see [[14], Chapter 8]. Then we have the pullback homomorphism $(\lambda_{\mathcal{L}_t})^* : H^1(J_t^\vee, \mathbb{Z}) \to H^1(J_t, \mathbb{Z})$ on cohomology groups induced by $\lambda_{\mathcal{L}_t}$ (see Definition 3.23).
From the above we get an injective homomorphism on cohomology groups ζ_t via the following commutative diagram

$$\begin{array}{ccc} H^1(A_t,\mathbb{Z}) & \xrightarrow{\zeta_t} & H^1(J_t,\mathbb{Z}) \\ \downarrow{\scriptstyle (\lambda_{\mathcal{L}'_t})_*} & & \uparrow{\scriptstyle (\lambda_{\mathcal{L}_t})^*} \\ H^1(A_t^\vee,\mathbb{Z}) & \xrightarrow{(i^\vee)^*} & H^1(J_t^\vee,\mathbb{Z}) \end{array}$$

Let $w_{t*} : H^1(J_t, \mathbb{Z}) \to H^1(C_t, \mathbb{Z})$ be the isomorphism given by Lemma 3.33 (Fact 3). By Proposition 14 in [1] the image of the composition $H^1(A_t, \mathbb{Z}) \overset{\zeta_t}{\to} H^1(J_t, \mathbb{Z}) \overset{w_{t*}}{\to} H^1(C_t, \mathbb{Z})$ is contained in the kernel of the Gysin homomorphism on cohomology groups $H^1(C_t, \mathbb{Z})_{\mathsf{van}} = \mathsf{Ker}(H^1(C_t, \mathbb{Z}) \overset{r_{t*}}{\to} H^3(S, \mathbb{Z}))$, i.e.,

$$(w_{t*} \circ \zeta_t)(H^1(A_t, \mathbb{Z})) \subset H^1(C_t, \mathbb{Z})_{\mathsf{van}}. \tag{2}$$

Now recall that B_t is the abelian subvariety of J_t corresponding to the Hodge substructure on $H^1(C_t, \mathbb{Z})_{\mathsf{van}}$, so $H^1(B_t, \mathbb{Z}) \cong (H^1(C_t, \mathbb{Z})_{\mathsf{van}})^* \cong H^1(C_t, \mathbb{Z})_{\mathsf{van}}$ (the composition of these isomorphisms is w_{t*}, see proof of Lemma 3.33). On the other hand, since w_{t*} is an isomorphism and ζ_t is injective we can identify $(w_{t*} \circ \zeta_t)(H^1(A_t, \mathbb{Z}))$ with $H^1(A_t, \mathbb{Z})$. Then by the inclusion (2), we get $H^1(A_t, \mathbb{Z}) \subset H^1(B_t, \mathbb{Z})$, so $A_t \subset B_t$. ☐

Remark 5.3. — Note that the proof of item a of Theorem 5.1 also holds for any smooth hyperplane section C of a surface S over and uncountable algebraically closed field k of characteristic zero and in the adequate context.

5.2. On the Proof of Item b. — In order to prove item b we will use the following lemmas. Let k be an uncountable algebraically closed field of characteristic 0. Let T be an integral scheme over k, $\mathscr{X}_T$ be a scheme over T and $X_t = f_T^{-1}(t)$ be the fiber over $t \in T$ of the flat family $f_T : \mathscr{X}_T \to T$. Recall that a c-open subset of an integral scheme is the complement of a c-closed subset (Definition 2.20).

Lemma 5.4. — *Given an integral base T over k there exits a natural c-open subset U_0 in T such that every $t \in U_0$ is scheme-theoretic isomorphic to the generic geometric point $\overline{\eta}$ point of T and given a flat family $f_T : \mathscr{X}_T \to T$ over T, the above scheme-theoretic isomorphism of points induce an isomorphism between the fiber X_t, for all closed points $t \in U_0$, and the geometric generic fiber $X_{\overline{\eta}}$, as schemes over $\mathsf{Spec}(\mathbb{Q})$, moreover these isomorphisms preserve rational equivalence of algebraic cycles (see* [1, §5]*).*

Proof. — Assume that T is affine (as one can always cover the integral scheme T by open affine subschemes).

Step 1. — We begin with the strategy of the construction of the c-open subset in T.
Recall that the transcendental degree $[k : \mathbb{Q}]$ of the uncountable algebraically closed field k over its primary subfield, i.e., over $\mathbb{Q}$, is infinity.
Since T is an integral affine scheme of finite type over k, then it is of the form $T = \mathsf{Spec}\left(\frac{k[x_1,\ldots,x_n]}{I(T)}\right)$, where $I(T) \subset k[x_1, \ldots, x_n]$ is the ideal of T.

Let $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ be a set of generators of $I(T)$. As the polynomials f_i have a finite number of coefficients, attaching the coefficients of $f_1, \dots, f_m$ to $\mathbb{Q}$ we get an extension of $\mathbb{Q}$, say $\widetilde{k}$, which is a countable subfield of k since $\mathbb{Q}$ is countable. Let k' be the algebraic closure of $\widetilde{k}$, then it is a countable algebraically closed subfield of k.
Let T' be the affine integral scheme defined by the ideal $I(T')$ generated by $f_1, \dots, f_m$ in $k'[x_1, \dots, x_n]$. Since $\frac{k[x_1,\dots,x_n]}{I(T)} = \frac{k'[x_1,\dots,x_n]}{I(T')} \otimes_{k'} k$ we have: $T = T' \times_{\mathsf{Spec}(k')} \mathsf{Spec}(k)$.
Denote by $k[T] = \frac{k[x_1,\dots,x_n]}{I(T)}$ (resp. $k[T'] = \frac{k'[x_1,\dots,x_n]}{I(T')}$) to the coordinate ring of T (resp. of T') and by $k(T)$ (resp. by $k(T')$) its function field.
Note that a closed subscheme Z' of T' is defined by an ideal $\mathfrak{a}$ in $k'[T'] = \frac{k'[x_1,\dots,x_n]}{I(T')}$, and let $i_{Z'} : Z' \hookrightarrow T'$ be the corresponding closed embedding. Since the field k' is countable and $\mathfrak{a}$ is finitely generated, there exist only countably many closed subschemes Z' in T'. For each Z' denote the complement by $U_{Z'} = T' \setminus \mathrm{im}(i_{Z'})$.
Let $Z = Z' \times_{\mathsf{Spec}(k')} \mathsf{Spec}(k), U_Z = U_{Z'} \times_{\mathsf{Spec}(k')} \mathsf{Spec}(k)$ and $i_Z : Z \hookrightarrow T$ be the pullbacks of $Z', U_{Z'}$ and $i_{Z'}$ respectively, with respect to the extension k/k', then $U_Z = T \setminus \mathrm{im}(i_Z)$.
Let $U_0 = T \setminus \bigcup_{Z'} \mathrm{im}(i_Z) = \bigcap_{Z'} U_Z$, where the union is taken over closed subschemes Z', such that $\mathrm{im}(i_Z) \neq T$. So U_0 is the complement of the countable union of Zariski closed subsets, i.e., U_0 is c-open by construction.
Recall that a k-point t of a scheme T is a section of the structural morphism $h : T \to \mathsf{Spec}(k)$, that is, a morphism $f_t : \mathsf{Spec}(k) \to T$ such that $h \circ f_t = id_{\mathsf{Spec}(k)}$.

Step 2. — Now we will see that there is an important isomorphism of fields related with each k-point of the c-open U_0 constructed above. More precisely

Claim. — Let $\overline{k(T)}$ be the algebraic closure of the field $k(T)$. For a k-point[1] t in U_0, one can construct a field isomorphism $e_t : \overline{k(T)} \xrightarrow{\sim} k$ such that for $f \in k'[T']$ we have $e_t(f) = f(t)$.

Proof of the Claim. — Let t be a k-point in U_0, that is, a morphism $f_t : \mathsf{Spec}(k) \to T$. Let $\pi : T = T' \times_{\mathsf{Spec}(k')} \mathsf{Spec}(k) \to T'$ be the projection, since $t \in U_0$ by the construction of U_0 we have that $\pi(t) = \eta' \in \bigcap_{Z'} U_{Z'}$ where the intersection is taken over the closed subschemes Z' of T' such that $\mathrm{im}(i_{Z'}) \neq T'$. Therefore η' is the generic point of T', since the generic point of an integral scheme is unique. This is the same to say that there exists a morphism $h_t : \{t\} = \mathsf{Spec}(k) \to \mathsf{Spec}(k'(T')) = \eta'$ such that the following diagram commutes

$$\begin{array}{ccc} \{t\} = \mathsf{Spec}(k) & \xrightarrow{f_t} & T \\ \downarrow{\scriptstyle h_t} & & \downarrow{\scriptstyle \pi} \\ \eta' = \mathsf{Spec}(k'(T')) & \xrightarrow{g'} & T' \end{array}$$

In terms of coordinate rings this means that there exist a homomorphism $\epsilon_t : k'(T') \to k$ such that the following diagram commutes

(3)
$$\begin{array}{ccc} k & \xleftarrow{ev_t} & k[T] \\ {\scriptstyle \epsilon_t}\uparrow & & \uparrow \\ k'(T') & \longleftarrow & k'[T'] \end{array}$$

[1]Since in our case k is algebraically closed, k-points of T coincide with closed points of T. So this claim is true for all closed points of U_0.

Here k is considered as the residue field of the scheme T at t, $ev_t : k[T] \to k$ is the evaluation at t morphism, corresponding to the morphism f_t, and that ϵ_t is the homomorphism corresponding to the morphism h_t.
Since $k'[T'] \to k[T]$ is injective, $k'[T'] \setminus \{0\}$ is a multiplicative system in $k[T]$. Furthermore we have $(k'[T'] \setminus \{0\})^{-1} k[T] = k[T] \otimes_{k'[T']} k'(T')$. Hence there exists a unique universal morphism $\varepsilon_t : k[T] \otimes_{k'[T']} k'(T') \to k$ such that $\varepsilon_t \mid_{k[T]} = ev_t$ and $\varepsilon_t \mid_{k'(T')} = \epsilon_t$.
We now construct an embedding of $k(T) \hookrightarrow k$ whose restriction to $k'(T')$ is ϵ_t.
Let $s = \dim(T') = \mathrm{Tr.deg}(k'(T)/k') =$ krull dimension of $k'[T']$. Here we denote by $\mathrm{Tr.deg}(k'(T)/k')$ to the transcendence degree of $k(T')$ over k', then by the Noether normalization lemma there exist s algebraically independent elements $x_1, \dots, x_s$ in $k'[T']$ such that $k'[T']$ is a finitely generated module over the polynomial ring $k'[x_1, \dots, x_s]$ and $k'(T')$ is algebraic over the field of fractions $k'(x_1, \dots, x_s)$.
It follows that $k[T]$ is a finitely generated module over the polynomial ring $k[x_1, \dots, x_s]$ and $k(T)$ is algebraic over the field of fractions $k(x_1, \dots, x_s)$.
Let $b_i = ev_t(x_i)$ for $i = 1, \dots, s$. Since $t \in U_0$ we have that $b_1, \dots, b_s$ are algebraically independent over k'. Indeed, if $b_1, \dots, b_s$ are algebraic dependent over k' there is a non-trivial polynomial f in s variables with coefficients in k' such that $f(b_1, \dots, b_s) = 0$ or equivalently such that $f(ev_t(x_1), \dots, ev_t(x_s)) = 0$, so we have a polynomial such that t is a zero of it, then $t \notin U_0$ which is a contradiction.
We can extend the set $b_1, \dots, b_s$ to a transcendental basis B of k over k', so that $k = k'(B)$. As B have an infinite cardinality $B \setminus \{b_1, \dots, b_s\}$ also have an infinite cardinality, choosing a bijection $B \xrightarrow{\sim} B \setminus \{b_1, \dots, b_s\}$ we obtain the following field embedding $k = k'(B) \simeq k'(B \setminus \{b_1, \dots, b_s\}) \subset k'(B)$ over k' such that $b_1, \dots, b_s$ is algebraically independent over its image. Then we get a field embedding $E_t : k(x_1, \dots, x_s) \hookrightarrow k$ sending x_i to b_i. Note that $E_t \mid_{k'(x_1,\dots,x_s)} = \epsilon_t \mid_{k'(x_1,\dots,x_s)}$.
Since $k(T) = k(x_1, \dots, x_s) \otimes_{k'(x_1,\dots,x_s)} k'(T')$, we get a uniquely defined embedding $k(T) \to k$ as the composition $k(T) \to k(x_1, \dots, x_s) \hookrightarrow k$. The embedding $k(T) \to k$ can extend to an isomorphism $e_t : \overline{k(T)} \xrightarrow{\sim} k$.
Finally, by the commutativity of the diagram (3), if we take $f \in k'[T']$ we can identify it with its image via the inclusions $k'[T'] \to k[T]$ and $k'[T'] \to k'(T')$, then we have $ev_t(f) = \epsilon_t(f)$. since $e_t|_{k'(T')} = \epsilon_t$ we have $e_t(f) = f(t)$. □

Step 3. — Given a $f_T : \mathscr{X}_T \to T$ a smooth morphism of schemes over k, we now see that the above isomorphism of fields induces an isomorphism of the fibers of f_T.
Let $f_T : \mathscr{X}_T \to T$ be a smooth morphism of schemes over k. Extending, if necessary, the field k' used to construct the c-open U_0 we may assume that there exists a morphism of schemes $f'_{T'} : \mathscr{X}'_{T'} \to T'$ over the countable algebraically closed field k', such that f_T is the pullback of $f'_{T'}$ under the field extension k/k'. Let

- $\eta' = \mathsf{Spec}(k'(T'))$ be the generic point of the affine scheme T', and let $X'_{\eta'}$ be the fibre of the family $f'_{T'} : \mathscr{X}'_{T'} \to T'$ over η',

- $\eta = \mathsf{Spec}(k(T))$ be the generic point of the affine scheme T, and let X_η be the fibre of the family $f_T : \mathscr{X}_T \to T$ over η,

- $\overline{\eta} = \mathsf{Spec}(\overline{k(T)})$ be the geometric generic point of the affine scheme T, and let $X_{\overline{\eta}}$ be the fibre of the family $f_T : \mathscr{X}_T \to T$ over $\overline{\eta}$.

The above isomorphism of fields $e_t : \overline{k(T)} \xrightarrow{\sim} k$, induces a scheme-theoretic isomorphism between the closed k-point $t \in U_0$ and the geometric generic point $\overline{\eta}$ of T

$$\mathsf{Spec}(e_t) : \{t\} = \mathsf{Spec}(k) \longrightarrow \mathsf{Spec}(\overline{k(T)}) = \{\overline{\eta}\}$$

over η', since we have $h_t = \mathsf{Spec}(\epsilon_t) : \mathsf{Spec}(k) \to \eta'$ and $\mathsf{Spec}(\overline{k(T)}) \to \eta'$.
Pulling back the scheme-theoretic isomorphism $\mathsf{Spec}(e_t)$ onto the fibres of the family f_T we obtain the cartesian squares

$$\begin{tikzcd} X_t \arrow[r, "\kappa_t"] \arrow[d] & X_{\overline{\eta}} \arrow[d] \\ \mathsf{Spec}(k) = \{t\} \arrow[r, "\mathsf{Spec}(e_t)"] & \mathsf{Spec}(\overline{k(T)}) = \overline{\eta} \end{tikzcd}$$

and pulling back $\mathsf{Spec}(\epsilon_t)$ onto the fibers we obtain

$$\begin{tikzcd} X_t \arrow[r] \arrow[d] & X'_{\eta'} \arrow[d] \\ \{t\} = \mathsf{Spec}(k) \arrow[r, "\mathsf{Spec}(\epsilon_t)"] & \mathsf{Spec}(k'(T')) = \eta' \end{tikzcd}$$

similarly, we get $X_{\overline{\eta}} \to X'_{\eta'}$ by pulling back $\mathsf{Spec}(\overline{k(T)}) \to \mathsf{Spec}(k'(T'))$.
Note that the morphism κ_t induced by $\mathsf{Spec}(e_t) = h_t$ is an isomorphism of schemes over $X'_{\eta'}$.

Step 4. — Next we describe the isomorphism between fibres X_t and $X_{t'}$ with $t, t' \in U_0$.
Let t' be another closed point of U_0, then we also have the isomorphism of fields $e_{t'} : \overline{k(T)} \xrightarrow{\sim} k(t') = k$, then $\sigma_{tt'} : k(t) = k \xrightarrow{e_t^{-1}} \overline{k(T)} \xrightarrow{e_{t'}} k = k(t')$ is an automorphism of k.
Let $(X_t)_{\sigma_{tt'}} = X_t \times_{\mathsf{Spec}(k)} \mathsf{Spec}(k)$ with respect to the automorphism of schemes $\mathsf{Spec}(\sigma_{tt'}) : \mathsf{Spec}(k) \to \mathsf{Spec}(k)$ induced by $\sigma_{tt'}$, and let $w_{\sigma_{tt'}} : (X_t)_{\sigma_{tt'}} \xrightarrow{\sim} X_t$ be the corresponding isomorphism on schemes over $\mathsf{Spec}(k^{\sigma_{tt'}})$, where $k^{\sigma_{tt'}}$ is a subfield of $\sigma_{tt'}$-invariants in k.
Since $k' \subset k^{\sigma_{tt'}} \subset k$, we have that the projection $\mathscr{X}_T \to \mathscr{X}'_{T'}$ factorises through $(\mathscr{X}'_{T'})_{k^{\sigma_{tt'}}} = \mathscr{X}'_{T'} \times_{\mathsf{Spec}(k')} \mathsf{Spec}(k^{\sigma_{tt'}})$, so, we can consider the fiber X_t as a scheme over $(\mathscr{X}'_{T'})_{k^{\sigma_{tt'}}}$ just by composing the inclusion of $X_t \hookrightarrow \mathscr{X}_T$ with the morphism $\mathscr{X}_T \to (\mathscr{X}'_{T'})_{k^{\sigma_{tt'}}}$.

Recall that $\sigma_{tt'} : k(t) = k \to k = k(t')$ and let $\kappa_{tt'} : X_{t'} \xrightarrow{\kappa_{t'}} X_{\overline{\eta}} \xrightarrow{\kappa_t^{-1}} X_t$ be the induced isomorphism of the fibres as schemes over $\mathsf{Spec}(k^{\sigma_{tt'}})$. It follows that: $(X_t)_{\sigma_{tt'}} = X_{t'}$, the isomorphism $w_{\sigma_{tt'}} : X_{t'} \xrightarrow{\sim} X_t$ is over $(\mathscr{X}'_{T'})_{k^{\sigma_{tt'}}}$, and $w_{\sigma_{tt'}} = \kappa_{tt'}$.

Step 5. — Finally, we have the following claim.

Claim. — The scheme-theoretic isomorphisms κ_t, for $t \in U_0$, preserve rational equivalence of algebraic cycles.

For the proof of this claim see [1, Lemma 19]. □

Now we study some facts about the closed embedded of the following two important family for us:

For any integral scheme T over $\mathbb{C}$ and for any morphism of schemes $T \to (\mathbb{P}^d)^*$ let $f_T : \mathscr{C}_T \to T$ be the family of hyperplane sections of S parametrized by T, $g_T : \mathcal{S}_T \to T$ the family such that each fiber over T is isomorphic to S, and

$$\begin{tikzcd} \mathscr{C}_T \arrow[rr, "r_T"] \arrow[dr, "f_T"'] & & \mathcal{S}_T \arrow[dl, "g_T"] \\ & T & \end{tikzcd}$$

the closed embedding of schemes over T. Then we also have closed embeddings r_t and $r_{\overline{\eta}}$ over $t = \mathsf{Spec}(\mathbb{C})$ and $\overline{\eta}$ respectively.

By Lemma 5.4 there exits a natural c-open subset U_0 in T such that the residue field of any closed point in U_0 is isomorphic to the residue field of the geometric generic point of T, since this isomorphism of fields induce a scheme-theoretic isomorphism between points, this c-open subset U_0 is such that any closed point $t \in U_0$ is scheme-theoretic isomorphic to the geometric generic point $\overline{\eta}$ of T.

Extending appropriately the countably algebraically closed field k', used to construct U_0, we may assume that there exists morphisms of schemes $f'_{T'}$, $g'_{T'}$ and $r'_{T'}$ over k' with $f'_{T'} = g'_{T'} \circ r'_{T'}$, and such that f_T, g_T and r_T are the pullback of $f'_{T'}$, $g'_{T'}$ and $r'_{T'}$ respectively under the field extension $\mathbb{C}/k'$. Then the scheme-theoretic isomorphism between the points $t \in U_0$ and the geometric generic point $\overline{\eta}$ of T induce isomorphisms $\kappa_t^{f_T} : C_t \to C_{\overline{\eta}}$ (resp. $\kappa_t^{g_T} : S_t \to S_{\overline{\eta}}$) between the fiber C_t (resp. S_t) over t and the geometric generic fiber $C_{\overline{\eta}}$ (resp. $S_{\overline{\eta}}$) over $\overline{\eta}$ of the family f_T (resp. g_T) for every $t \in U_0$, as schemes over $\mathsf{Spec}(\mathbb{Q})$, and for any two points t and t' in U_0 one has the isomorphisms $\kappa_{tt'}^{f_T} : C_t \to C_{t'}$ (resp. $\kappa_{tt'}^{g_T} : S_t \to S_{t'}$). Moreover, for any closed point $t \in U_0$, the following diagram

$$\begin{tikzcd} C_t \arrow[r, "r_t"] \arrow[d, "\kappa_t^{f_T}"] & S_t \arrow[d, "\kappa_t^{g_T}"] \\ C_{\overline{\eta}} \arrow[r, "r_{\overline{\eta}}"] & S_{\overline{\eta}} \end{tikzcd} \tag{4}$$

commutes, where r_t and $r_{\overline{\eta}}$ are the morphisms on fibres induced by r_T. Then the isomorphisms $\kappa_{tt'}^{f_T} = (\kappa_t^{f_T})^{-1} \circ \kappa_{t'}^{f_T}$ (resp. $\kappa_{tt'}^{g_T} = (\kappa_t^{g_T})^{-1} \circ \kappa_{t'}^{g_T}$) commute with closed embeddings r_t and $r_{t'}$ for any two closed points t, t' in U_0. Removing more Zariski closed subset from U_0 if necessary we may assume that the fibres of the families f_T and g_T over the points on U_0 are smooth, that is, we can assume that $U_0 \subset U$.

For every closed point $t \in U_0$, let $alb_{C_t} : \mathsf{CH}_0(C_t)_{\deg=0} \xrightarrow{\sim} J_t$ be the corresponding isomorphisms given by Fact 1 (see Lemma 3.46) and denote by $alb_{C_{\overline{\eta}}} : \mathsf{CH}_0(C_{\overline{\eta}})_{\deg=0} \xrightarrow{\sim} J_{\overline{\eta}}$ to the isomorphism for the geometric generic fiber (see Remark 3.47).

By the Step 5 of Lemma 5.4, for any $t \in U_0$ the scheme-theoretic isomorphisms $\kappa_t^{f_T} : C_t \to C_{\overline{\eta}}$ of the family f_T preserve rational equivalence, then they induce the push-forward isomorphisms of Chow groups $\kappa_{t*}^{f_T} : \mathsf{CH}_0(C_t)_{\deg=0} \to \mathsf{CH}_0(C_{\overline{\eta}})_{\deg=0}$ then we get $l_t : J_t \to J_{\overline{\eta}}$ as the composition given by the commutative diagram

$$\begin{tikzcd}
\mathsf{CH}_0(C_t)_{\deg=0} \arrow[r, "\kappa_{t*}^{f_T}"] & \mathsf{CH}_0(C_{\overline{\eta}})_{\deg=0} \arrow[d, "alb_{C_{\overline{\eta}}}"] \\
J_t \arrow[u, "alb_{C_t}^{-1}"] \arrow[r, "l_t"] & J_{\overline{\eta}}
\end{tikzcd}$$

Now consider the following commutative diagram

$$\begin{tikzcd}
\mathsf{Sym}^{d,d}(C_t) \arrow[r, "\mathsf{Sym}^{d,d}(\kappa_t^{f_T})"] \arrow[d, "\theta_d^{C_t}"] & \mathsf{Sym}^{d,d}(C_{\overline{\eta}}) \arrow[d, "\theta_d^{C_{\overline{\eta}}}"] \\
\mathsf{CH}_0(C_t)_{\deg=0} \arrow[r, "\kappa_{t*}^{f_T}"] \arrow[d, "alb_{C_t}"] & \mathsf{CH}_0(C_{\overline{\eta}})_{\deg=0} \arrow[d, "alb_{C_{\overline{\eta}}}"] \\
J_t \arrow[r, "l_t"] & J_{\overline{\eta}}
\end{tikzcd}$$

Since $alb_{C_t} \circ \theta_d^{C_t}$ is a regular morphism of schemes over $\mathbb{C}$ and $alb_{C_{\overline{\eta}}} \circ \theta_d^{C_{\overline{\eta}}}$ is a regular morphism of schemes over $\overline{\mathbb{C}(T)}$ the algebraic closure of the function field of T (see Lemma 2.27 and Lemma 2.29) and the morphism $\mathsf{Sym}^{d,d}(\kappa_t^{f_T})$ is a regular morphism over $\mathbb{Q}$, it follows that the homomorphism $l_t : J_t \to J_{\overline{\eta}}$ is a regular morphism of schemes over $\mathbb{Q}$.
Similarly, by Step 5 of Lemma 5.4, for any $t \in U_0$ the scheme-theoretic isomorphisms $\kappa_t^{g_T} : S_t \to S_{\overline{\eta}}$ on the fibers of the family g_T preserve rational equivalence, then they induce the push-forward isomorphisms of groups $\kappa_{t*}^{g_T} : \mathsf{CH}_0(S_t)_{\deg=0} \to \mathsf{CH}_0(S_{\overline{\eta}})_{\deg=0}$, and from the commutative diagram (4) one obtains the commutative diagram in Chow groups

$$\begin{tikzcd}
\mathsf{CH}_0(C_t)_{\deg=0} \arrow[r, "r_{t*}"] \arrow[d, "\kappa_{t*}^{f_T}"] & \mathsf{CH}_0(S_t)_{\deg=0} \arrow[d, "\kappa_{t*}^{g_T}"] \\
\mathsf{CH}_0(C_{\overline{\eta}})_{\deg=0} \arrow[r, "r_{\overline{\eta}*}"] & \mathsf{CH}_0(S_{\overline{\eta}})_{\deg=0}
\end{tikzcd} \tag{5}$$

For every closed $t \in U_0$, let A_t and B_t be the abelian subvarieties of J_t obtained in the proof of item a and let $A_{\overline{\eta}}$ and $B_{\overline{\eta}}$ be the abelian subvarieties of $J_{\overline{\eta}}$ which can be obtained in a similar way to the proof of item a corresponding to the closed embedding $r_{\overline{\eta}} : C_{\overline{\eta}} \to S_{\overline{\eta}}$ (see Remark 5.3).

Lemma 5.5. — *For any closed point $t \in U_0$, $l_t(B_t) = B_{\overline{\eta}}$ and $l_t(A_t) = A_{\overline{\eta}}$.*

Proof. — To prove $l_t(A_t) = A_{\overline{\eta}}$ recall that by item a we have $G_t = \bigcup_{x \in \Xi_t}(x + A_t)$ and $G_{\overline{\eta}} = \bigcup_{x \in \Xi_{\overline{\eta}}}(x + A_{\overline{\eta}})$. By definition $l_t = alb_{C_{\overline{\eta}}} \circ \kappa_{t*}^{f_T} \circ alb_{C_t}^{-1}$ then $l_t(G_t) = alb_{C_{\overline{\eta}}} \circ \kappa_{t*}^{f_T} \circ alb_{C_t}^{-1}(G_t)$ equivalently we have $l_t(G_t) = alb_{C_{\overline{\eta}}} \circ \kappa_{t*}^{f_T}(G_t)$ via the identification $alb_{C_t}^{-1}$. By the commutative diagram (5) we have $\kappa_{t*}^{f_T}(G_t) = G_{\overline{\eta}}$, then

$$l_t(G_t) = G_{\overline{\eta}} \tag{6}$$

via the isomorphism $alb_{C_{\overline{\eta}}}$. On the other hand,

$$l_t(G_t) = l_t\left(\bigcup_{x\in\Xi_t}(x+A_t)\right) = \bigcup_{x\in\Xi_t}(l_t(x)+l_t(A_t)) \tag{7}$$

By equations (6) and (7) we have $\bigcup_{x\in\Xi_t}(l_t(x)+l_t(A_t)) = \bigcup_{x\in\Xi_{\overline{\eta}}}(x+A_{\overline{\eta}})$ inside of $J_{\overline{\eta}}$. Note that $l_t(A_t)$ is Zariski closed in $J_{\overline{\eta}}$ since the group isomorphism l_t are regular morphisms of schemes over $\mathsf{Spec}(\mathbb{Q})$. Since $l_t(A_t)$ is a subgroup of in $J_{\overline{\eta}}$, it is an abelian subvariety in $J_{\overline{\eta}}$. As the right and left terms of the above equality are irredundant decomposition of $G_{\overline{\eta}}$ by the uniqueness of it (see Lemma 2.18) and by the fact that the irredundant decomposition of $G_{\overline{\eta}}$ must contain a unique irreducible component passing through 0 (see Lemma 2.19) we have that $l(A_t) = A_{\overline{\eta}}$. □

Remark 5.6. — The above Lemma 5.5 tells us that one can study the varieties A_t in a family either working at the geometric generic point or at a very general closed point on the base scheme.

Now, choose $L \cong \mathbb{P}^1$ be a Lefschetz pencil of hyperplanes for the surface S (see Definition 4.6 and Proposition 4.10) such that $L \cap U_0 \neq \emptyset$.

Lemma 5.7. — *Let $t \in L \cap U_0$, let A_t be the abelian subvariety of $B_t \subset J_t$ obtained in the proof of item a). Then either $A_t = 0$ or $A_t = B_t$.*

Proof. — Since $L \cong \mathbb{P}^1$ be a Lefschetz pencil of hyperplanes for the surface S passing through t (if we think of this Lefschetz pencil as the family of hyperplane sections $(C_t)_{t\in L}$ parametrized by L this means that C_t corresponding to this t is a member of this family), then it gives rise to a morphism $f_L : \mathscr{C}_L \to L$, where $\mathscr{C}_L$ is smooth because it can be identified with the blowing up $\widetilde{S} = \{(x,t) \in S \times L : x \in C_t = S \cap H_t\}$ of S at the base locus A_L of the pencil, and $f_L = \mathsf{pr}_2|_{\tilde{S}} : \widetilde{S} \to L$. Moreover, each hyperplane section C_t of S parametrized by points of L can be naturally identified with the fibre $f_L^{-1}(t) \subset \widetilde{S}$, so each fibre C_t of f_L has at most one ordinary double point as singularity.
Suppose in addition that $\dim(\Delta_S) = d-1$, i.e., the discriminant locus is a hypersurface, then by Proposition 4.10 L meets the discriminant hypersurface Δ_S transversely in the open dense subset $\Delta_S^0 \subset \Delta_S$ parametrizing hyperplanes in $\mathbb{P}^d$ such that the corresponding hyperplane sections of S has at most one ordinary double point as singularity, that is $L\cap\Delta_S = \{0_1, \dots, 0_M\}$ is a finite subset of L, then from Remark 4.53 and Zariski Theorem 4.54 we can conclude that if we denote by $V = L - L \cap \Delta_S$, then we have the monodromy action

$$\rho_V : \pi_1(V,t) \longrightarrow \mathsf{Aut}(H^1(C_t,\mathbb{Q}))_{\mathsf{van}}.$$

Now we claim that the local monodromy representation ρ_V is irreducible.

Proof of the claim. — Indeed, recall that $L \cap \Delta_S = \{0_1, \dots, 0_M\}$ are the critical values of the Lefschetz pencil $L \cong \mathbb{P}^1$. For each 0_i with $i = 1, \dots, M$, consider the small disk $D_i \subset L$ centered at 0_i, $t_i \in D_i^*$ and γ_i the path joining t to t_i. Let $\delta_i' \in H_1(C_{t_i},\mathbb{Q}) = H^1(C_{t_i},\mathbb{Q})$ be the vanishing cycle (the homology class of the vanishing sphere $S^1_{t_i} \subset C_{t_i}$) which is well defined up to sign as a generator of $\mathsf{Ker}(H^1(C_{t_i},\mathbb{Q}) \to H^1(C_{\Delta_i},\mathbb{Q}))$ and recall that by trivialising the fibration $f_L|_{\gamma_i}$ over γ_i we can construct a diffeomorphism $C_{t_i} \cong C_t$, so we have a vanishing

cycle $\delta_i \in H^1(C_t,\mathbb{Q})$ which is image of δ_i' via the diffeomorphism. By Lemma 4.30 the vanishing cohomology $H^1(C_t,\mathbb{Q})_{\mathsf{van}}$ is generated by these vanishing cycles δ_i, $i=1,\dots,M$, of the Lefschetz pencil L.
On the other hand, let $\widetilde{\gamma}_i$ be the loop in V based at t such that $\widetilde{\gamma}_i$ is equal to γ_i until t_i winds around the disk D_i once in the positive direction, and then returns to t via γ_i^{-1}. Recall that these loops $\widetilde{\gamma}_i, i=1,\dots,M$, generate $\pi_1(V,t)$ and note that the image of the loops $\widetilde{\gamma}_i$ via ρ_V are elements in $\mathsf{Aut}(H^1(C_t,\mathbb{Q})_{\mathsf{van}})$.
Let $F\subset H^1(C_t,\mathbb{Q})_{\mathsf{van}}$ be a nontrivial vector subspace which is stable under the monodromy action ρ_V. We must prove that $F=H^1(C_t,\mathbb{Q})_{\mathsf{van}}$.
Let $0\neq\alpha\in F$. Since by Proposition 4.32, $\langle\cdot,\cdot\rangle$ is non-degenerate on $H^1(C_t,\mathbb{Q})_{\mathsf{van}}$ there exists $i\in\{1,\dots,M\}$ such that $\langle\alpha,\delta_i\rangle\neq 0$.
By the Picard–Lefschetz formula (Theorem 4.47) for $\alpha\in F\subset H^1(C_t,\mathbb{Q})_{\mathsf{van}}$ one has $\rho_V(\widetilde{\gamma}_i)(\alpha)=\alpha\pm\langle\alpha,\delta_i\rangle\delta_i$ or equivalently, $\rho_V(\widetilde{\gamma}_i)(\alpha)-\alpha=\pm\langle\alpha,\delta_i\rangle\delta_i$.
Since, by assumption, F is a vector subspace of $H^1(C_t,\mathbb{Q})_{\mathsf{van}}$ which is stable under the monodromy action ρ_V (so $\rho_V(\widetilde{\gamma}_i)(\alpha)\in F$) we have $\rho_V(\widetilde{\gamma}_i)(\alpha)-\alpha=\pm\langle\alpha,\delta_i\rangle\delta_i\in F$. Then $\delta_i\in F$.
But, Corollary 4.56 shows that all the vanishing cycles are conjugate under the monodromy action, so F, which is stable under the monodromy action, must contain all the vanishing cycles. Thus $F=H^1(C_t,\mathbb{Q})_{\mathsf{van}}$. □

On the other hand, by Lemma 3.33 (Fact 3) we have $H^1(C_t,\mathbb{Z})\overset{w_{t*}}{\simeq}H^1(J_t,\mathbb{Z})$ is an isomorphism. Let $H^1(C_t,\mathbb{Q})=H^1(C_t,\mathbb{Z})\otimes\mathbb{Q}\overset{(w_{t*})_{\mathbb{Q}}}{\simeq}H^1(J_t,\mathbb{Z})\otimes\mathbb{Q}=H^1(J_t,\mathbb{Q})$ be the isomorphism induced by w_{t*}, we get this because in particular C_t is compact (see [22, §7.1.1]).
Let $L_t=(w_{t*})_{\mathbb{Q}}^{-1}(H^1(A_t,\mathbb{Q}))$ be the (pre)image in $H^1(C_t,\mathbb{Q})$ of $H^1(A_t,\mathbb{Q})\subset H^1(J_t,\mathbb{Q})$ under the isomorphism $(w_{t*})_{\mathbb{Q}}^{-1}$. Then L_t is a $\mathbb{Q}$-vector subspace in $H^1(C_t,\mathbb{Q})$.
Recall also that $H^1(B_t,\mathbb{Z})\overset{w_{t*}}{\simeq}H^1(C_t,\mathbb{Z})_{\mathsf{van}}$ (see final part of the proof of item a)), then it follows that $H^1(B_t,\mathbb{Q})\overset{(w_{t*})_{\mathbb{Q}}}{\simeq}H^1(C_t,\mathbb{Q})_{\mathsf{van}}$.
Since in item a) we proved that $H^1(A_t,\mathbb{Z})\subset H^1(B_t,\mathbb{Z})$ we get $H^1(A_t,\mathbb{Q})\subset H^1(B_t,\mathbb{Q})$, this implies that $L_t\subset H^1(C_t,\mathbb{Q})_{\mathsf{van}}$ in $H^1(C_t,\mathbb{Q})$ via the isomorphism $(w_{t*})_{\mathbb{Q}}$. Moreover, L_t is a vector subspace in $H^1(C_t,\mathbb{Q})_{\mathsf{van}}$ which has a Hodge structure on it since it corresponds to the abelian subvariety A_t of J_t, then it is invariant under the monodromy representation ρ_V (see Proposition 4.44).
Then, since the monodromy action $\rho_V:\pi_1(V,t)\to\mathsf{Aut}(H^1(C_t,\mathbb{Q})_{\mathsf{van}})$ on $H^1(C_t,\mathbb{Q})_{\mathsf{van}}$ is irreducible, either $L_t\overset{(w_{t*})_{\mathbb{Q}}}{\simeq}H^1(A_t,\mathbb{Q})=0$ and then $A_t=0$, or $L_t\overset{(w_{t*})_{\mathbb{Q}}}{\simeq}H^1(A_t,\mathbb{Q})=H^1(C_t,\mathbb{Q})_{\mathsf{van}}\overset{(w_{t*})_{\mathbb{Q}}}{\simeq}H^1(B_t,\mathbb{Q})$ and then $A_t=B_t$. □

Finally, we next prove the item b of the main result of this paper.

Proof of item b of Theorem 5.1. — Let $f:\mathscr{C}\to\mathbb{P}^{d*}$ be the universal hyperplane section of S, i.e., the family of hyperplane sections of S parametrized by $\mathbb{P}^{d*}$ (see Example 4.51). Let $g:\mathcal{S}=S\times\mathbb{P}^{d*}\to\mathbb{P}^{d*}$ be the trivial family parametrised by $\mathbb{P}^{d*}$, i.e., the family such that each fiber over $\mathbb{P}^{d*}$ is isomorphic to S.

Let

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "r"] \arrow[dr, "f"'] & & \mathcal{S} \arrow[dl, "g"] \\ & \mathbb{P}^{d*} & \end{tikzcd}$$

is the closed embedding of schemes over $\mathbb{P}^{d*}$.

By Lemma 5.4 there exits a natural c-open subset U_0 in $\mathbb{P}^{d*}$ such that the residue field of any closed point in U_0 is isomorphic to the residue field of the geometric generic point of $\mathbb{P}^{d*}$, since this isomorphism of fields induces a scheme-theoretic isomorphism between points, this c-open U_0 is such that any closed point $t \in U_0$ is scheme-theoretic isomorphic to the geometric generic point $\overline{\eta}$ of $\mathbb{P}^{d*}$.

Extending appropriately the countably algebraically closed field $k' \subset \mathbb{C}$, used to construct U_0, we may assume that there exists morphisms of schemes f', g' and r' over k' with $f' = g' \circ r'$ and such that f, g and r are the pullback of f', g' and r' respectively under the field extension $\mathbb{C}/k'$. Then the scheme-theoretic isomorphism between the points $t \in U_0$ and the geometric generic point $\overline{\eta}$ of $\mathbb{P}^{d*}$ induces isomorphisms κ_t^f (resp. κ_t^g) between the fiber C_t (resp. S_t) over t and the geometric generic fiber $C_{\overline{\eta}}$ (resp. $S_{\overline{\eta}}$) over $\overline{\eta}$ of the family f (resp. g) for every $t \in U_0$, as schemes over $\mathsf{Spec}(\mathbb{Q})$, and for any two points t and t' in U_0 one has isomorphisms $\kappa_{tt'}^f : C_t \to C_{t'}$ (resp. $\kappa_{tt'}^g : S_t \to S_{t'}$). Moreover, for any closed point $t \in U_0$, the following diagram

$$\begin{tikzcd} C_t \arrow[r, "r_t"] \arrow[d, "\kappa_t^f"] & S_t \arrow[d, "\kappa_t^g"] \\ C_{\overline{\eta}} \arrow[r, "r_{\overline{\eta}}"] & S_{\overline{\eta}} \end{tikzcd}$$

commutes, where r_t and $r_{\overline{\eta}}$ are the morphisms on fibres induced by r. Then the isomorphisms $\kappa_{tt'}^f = (\kappa_t^f)^{-1} \circ \kappa_{t'}^f$ (resp. $\kappa_{tt'}^g = (\kappa_t^g)^{-1} \circ \kappa_{t'}^g$) commute with closed embeddings r_t and $r_{t'}$ for any two closed points t, t' in U_0. Removing more Zariski closed subset from U_0 if necessary we may assume that the fibres of the families f and g over the points on U_0 are smooth, that is, we can assume that $U_0 \subset U$.

Recall that for every closed point $t \in \mathbb{P}^{d*}$ we denote by H_t the corresponding hyperplane in $\mathbb{P}^d$.

Let $\Omega \subset \mathbb{P}^{d*}$ be a Zariski closed subset in $\mathbb{P}^{d*}$ such that for every point in $t \in \mathbb{P}^{d*} - \Omega$ the corresponding hyperplane H_t does not contain S and $H_t \cap S = C_t$ is either smooth or contains at most one singular point which is a double point.

Let $G(1, \mathbb{P}^{d*})$ be the Grassmannian of lines in $\mathbb{P}^{d*}$. There exists $W \subset G(1, \mathbb{P}^{d*})$ a Zariski open subset of $G(1, \mathbb{P}^{d*})$ such that for every line $L \in W$ we have $L \cap \Omega = \emptyset$ and its corresponding codimension 2 linear subspace A_L in $\mathbb{P}^d$ intersects S transversally. In other words, any line $L \in W$ gives rise to a Lefschetz pencil for S (see Corollary 4.12).

Let $Z = \mathbb{P}^{d*} - U_0$ be the complement of the c-open U_0 subset of $\mathbb{P}^{d*}$, then Z is c-closed. It follows that the condition for a line $L \in G(1, \mathbb{P}^{d*})$ to be not a subset in Z is c-open. This means that there exists a c-open $A \subset G(1, \mathbb{P}^{d*})$ such that for $L \in A$ we have $L \not\subset Z$. It follows that $A \cap W \neq \emptyset$, so we can choose a line $L \subset \mathbb{P}^{d*}$ such that it gives rise to a Lefschetz pencil $f_L : \mathscr{C}_L \to L$ for S and $L \cap U_0 \neq \emptyset$.

Let $t_0 \in L \cap U_0$, then by Lemma 5.7 $A_{t_0} = 0$ or $A_{t_0} = B_{t_0}$.

Suppose that $A_{t_0} = 0$. Applying the Lemma 5.5 to the case $T = \mathbb{P}^{d*}$, we obtain $A_{\overline{\eta}} = 0$ because t_0 and $\overline{\eta}$ are isomorphic since $t_0 \in U_0$. Then applying the same Lemma 5.5 we have $A_t = 0$ for each closed point $t \in U_0$.
Suppose that $A_{t_0} = B_{t_0}$. Applying the Lemma 5.5 to the case $T = \mathbb{P}^{d*}$, we obtain $A_{\overline{\eta}} = B_{\overline{\eta}}$ because t_0 and $\overline{\eta}$ are isomorphic since $t_0 \in U_0$. Then applying the same Lemma 5.5 we have $A_t = B_t$ for each closed point $t \in U_0$. $\square$

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