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THE *abcd* CONJECTURE, UNIFORM BOUNDEDNESS, AND DYNAMICAL SYSTEMS

by

Robin Zhang

To the memory of Lucien Szpiro (1941–2020)

Abstract. — We survey Vojta's higher-dimensional generalizations of the *abc* conjecture and Szpiro's conjecture as well as recent developments that apply them to various problems in arithmetic dynamics. In particular, the "*abcd* conjecture" implies a dynamical analogue of a conjecture on the uniform boundedness of torsion points and a dynamical analogue of Lang's conjecture on lower bounds for canonical heights.

Résumé. — Nous décrivons des généralisations en dimension supérieure dues à Vojta de la conjecture *abc* et de la conjecture de Szpiro, ainsi que des avancées récentes qui les utilisent dans des problèmes variés de dynamique arithmétique. En particulier, la « conjecture *abcd* » implique un analogue dynamique de la conjecture de torsion et un analogue dynamique de la conjecture de Lang sur les minorations de hauteurs canoniques.

1. Szpiro to *abc*

The *abc* conjecture, which originated in a conversation between Masser and Oesterlé in 1985 as an approach to Szpiro's conjecture (cf. [52, Section 3]), has been popularly described as "the most important unsolved problem in Diophantine analysis" [18]. Many articles have been written about their implications across number theory and Diophantine geometry (e.g. the wonderful surveys [18, 19, 23, 68]).

This article highlights recent developments that extend this discussion, with applications of the "*abcd* conjecture" to various problems in arithmetic dynamics. This *abcd* conjecture lies between Vojta's higher-dimensional "*abcde*... conjecture" and the classical *abc* conjecture. It has been shown by Looper [38, 39] to imply the uniform boundedness of preperiodic points of polynomials on $\mathbb{P}^1$ and a weak version of the dynamical Lang conjecture on points of small canonical height. The conditional uniform boundedness of preperiodic points also yields the conditional non-existence of rational (or even quadratic) periodic points of unicritical polynomials of large enough degree.

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1.1. Szpiro's conjecture. — Let E be an elliptic curve over the rational numbers $\mathbb{Q}$ with global minimal Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Tate's notes [64] (based on his letter to Cassels) define invariants of E,

$$\begin{aligned} c_2 &= a_1^2 + 4a_2, \\ c_4 &= a_1a_3 + 2a_4, \\ c_6 &= a_3^2 + 4a_6, \\ c_8 &= a_1^2a_6 - a_1a_3a_4 + 4a_2a_6 + a_2a_3^2 - a_4^2. \end{aligned}$$

The discriminant Δ_E is $-c_2^2c_8 - 8c_4^3 - 27c_6^2 + 9c_2c_4c_6$. This discriminant is minimal by our minimality assumption on the model. The conductor N_E of E is the product

$$N_E := \prod_{p \text{ prime}} p^{\epsilon_p + \delta_p}.$$

The tame part ϵ_p of the conductor is

$$\epsilon_p := \begin{cases} 0 & \text{if } E \text{ has good reduction at } p, \\ 1 & \text{if } E \text{ has multiplicative reduction at } p, \\ 2 & \text{if } E \text{ has additive reduction at } p. \end{cases}$$

The wild part δ_p comes from the ℓ-adic Swan representation and is nonzero only if $p < 5$ (for more about the conductor of elliptic curves, see Silverman [60, Section IV.10]).
Following Szpiro's work on the Shafarevich and Mordell conjectures over function fields of positive characteristic, he hoped to use Arakelov theory to prove the effective Mordell conjecture (see [15, p. 1771–1774]). Szpiro's conjecture was presented at a talk in Hannover in 1982 in the following form.

Conjecture 1.1 **(Weak Szpiro's conjecture** [52, Conjecture 1]**).** — *There exist positive real numbers α and β such that*

$$|\Delta_E| \leq \alpha N_E^{\beta},$$

for any elliptic curve E over $\mathbb{Q}$ with minimal discriminant Δ_E and conductor N_E.

With a lower bound on β and with an expression of α as a function of β, there are stronger versions of Szpiro's conjecture.

Conjecture 1.2 **(Strong Szpiro's conjecture** [52, Conjecture 2]**).** — *For all $\epsilon > 0$, there is a constant $C(\epsilon) > 0$ such that*

$$|\Delta_E| \leq C(\epsilon) N_E^{6+\epsilon},$$

for any elliptic curve E over $\mathbb{Q}$ with minimal discriminant Δ_E and conductor N_E.

Conjecture 1.3 **(Modified Szpiro's conjecture** [52, Conjecture 4]**).** — *For each $\epsilon > 0$, there is a constant $C(\epsilon) > 0$ such that*

$$\max\left\{|c_4|^3, |c_6|^2\right\} \leq C(\epsilon) N_E^{6+\epsilon},$$

for any elliptic curve E over $\mathbb{Q}$ with invariants c_4, c_6, and conductor N_E.

The many arithmetic implications of Szpiro's conjecture (and its explicit variations) famously include the Mordell conjecture [14], Fermat's Last Theorem [23], Baker's theorem [23], Roth's theorem on Diophantine approximations [23], Lang's height conjecture [24, p. 420], and the non-existence of Siegel zeroes for certain L-functions [22, Theorem 2].

Remark 1.4. — It is worth noting the significant attention that Szpiro's conjecture has received since the 2012 release of Shinichi Mochizuki's four preprints claiming a proof via inter-universal Teichmüller theory. These four papers have since been published [44, 45, 46, 47] with an additional follow-up article [49], although the academic disagreements have not yet been completely resolved to the author's knowledge (c.f. [9, 13, 32, 48, 58]).

1.2. ***abc*** **conjecture.** — Seeking to formulate Szpiro's conjecture without referring to elliptic curves, Masser and Oesterlé formulated the *abc* conjecture in 1985 (cf. [52, Section 3]).

Conjecture 1.5 **(The** ***abc*** **conjecture** [52, Conjecture 3]**).** — *For every positive real number ϵ, there exists a positive real number $C(\epsilon)$ such that*

$$c < C(\epsilon)\,\mathrm{rad}(abc)^{1+\epsilon},$$

for every triple (a, b, c) of coprime positive integers such that $a + b = c$.

The *abc* conjecture was then shown to be equivalent to the modified Szpiro's conjecture by Oesterlé and Nitaj (cf. [52, Section 3], [19, Section 4], [15, p. 1769]). One way to see that Szpiro's conjecture implies a weak *abc* conjecture is to take the Hellegouarch–Frey curve

$$E_{a,b} : y^2 = x(x-a)(x-b),$$

for any positive coprime integers (a, b, c) such that $a + b = c$. The elliptic curve $E_{a,b}$, after taking a minimal model, has minimal discriminant $\Delta_{a,b} = 2^{-s}(abc)^2$ and conductor $N_{a,b} = 2^{-t}\,\mathrm{rad}(abc)$ where s and t are bounded integers. If Conjecture 1.3 holds for $E_{a,b}$, then for all $\epsilon > 0$, there is a constant $C(\epsilon) > 0$ such that

$$\begin{aligned}
|\Delta_{a,b}| &\leq C(\epsilon) N_{a,b}^{6+\epsilon},\\
\left|2^{-s}(abc)^2\right| &\leq C(\epsilon)\left(2^{-t}\,\mathrm{rad}(abc)\right)^{6+\epsilon},\\
(abc)^2 &\leq C(\epsilon)\frac{\left(2^{-t}\right)^{6+\epsilon}}{|2^{-s}|}\,\mathrm{rad}(abc)^{6+\epsilon}.
\end{aligned}$$

Folding $\frac{\left(2^{-t}\right)^{6+\epsilon}}{|2^{-s}|}$ into $C(\epsilon)$ and observing that $c = a + b$ is greater than a and b, we obtain

$$c^4 \leq (abc)^2 \leq C(\epsilon)\left(\mathrm{rad}(abc)\right)^{6+\epsilon} \leq C(\epsilon)\,\mathrm{rad}(abc)^{6+6\epsilon},$$

and thus,

$$c \leq C(\epsilon)\,\mathrm{rad}(abc)^{\frac{3}{2}+\epsilon}.$$

Remark 1.6. — In fact, Baker [1] and Laishram–Shorey [33] give empirical evidence for an explicit version of Conjecture 1.5 with $\epsilon = \frac{3}{4}$ and $C\left(\frac{3}{4}\right) \leq 1$ (these exponents can be even further optimized, cf. [11]).

2. *abcd*... conjectures

2.1. Background and notation. — There are generalizations of the *abc* conjecture in various directions, such as the uniform version over general number fields given by Vojta [65]. Vojta [66] also proposed a higher-dimensional generalization over general number fields using the arithmetic truncated counting function in the language of Arakelov theory. We try to present the language of [66] (e.g. using the arithmetic model) as well as the language of [38, 39] (e.g. using local height functions).

Let K be a number field or one-variable function field of characteristic 0 with set of places M_K. If K is a function field, then $K = k(\mathbb{M}_K)$ (the function field of $\mathbb{M}_K$) for some smooth curve $\mathbb{M}_K$ over a field of constants $k_0 \subset K$ and there is a bijection between M_K and the closed points of $\mathbb{M}_K$. If K is a number field, then there is an arithmetic scheme $\mathbb{M}_K$ whose closed points are also in bijection with M_K and whose non-archimedean part $(\mathbb{M}_K)_{\mathrm{na}}$ is just Spec $\mathscr{O}_K$. An arithmetic model $\mathscr{X}$ of X is an integral, flat, separated scheme $\mathscr{X}_{\mathrm{na}}$ of finite type over $(\mathbb{M}_K)_{\mathrm{na}}$ (plus additional archimedean information) with an isomorphism $X \cong \mathscr{X}_{\mathrm{na}} \times_{(\mathbb{M}_K)_{\mathrm{na}}} K$.

Remark 2.1. — In this article, we will largely ignore the Green functions and archimedean phenomena (cf. [66]).

For a place $\nu \in M_K$, denote the residue field of K at ν by k_ν. Define μ_ν and r_ν via

$$\mu_\nu := \begin{cases} \log(\# k_\nu) & \text{if } K \text{ is a number field,} \\ [k_\nu : k_0] & \text{if } K \text{ is a function field.} \end{cases}$$

$$r_\nu := \begin{cases} \frac{[K_\nu:\mathbb{Q}_\nu]}{[K:\mathbb{Q}]} & \text{if } K \text{ is a number field,} \\ \frac{[K_\nu:k_0(t)_\nu]}{[K:k_0(t)]} & \text{if } K \text{ is a function field.} \end{cases}$$

For a Cartier divisor D on $\mathbb{M}_K$, let $n_\nu(D)$ be the multiplicity of D at the closed point of $\mathbb{M}_K$ corresponding to ν.

Relative to a divisor D of a smooth complete variety X over a field K and ν-adic local heights (i.e. local Weil functions) $\lambda_{D,\nu}$ (cf. [65, Section 1.3], [26, Section B.8]), we will denote the height of a point $P \in X(\overline{K}) \setminus \mathrm{Supp}(D)$ by

$$h_D(P) := \sum_{\nu \in M_K} r_\nu \lambda_{D,\nu}(P).$$

Alternatively, the height can be given as

$$h_D(P) = \frac{\deg \sigma^* D}{[L : K]}$$

where D is a Cartier divisor on $\mathscr{X}$, P is not in the support of D, E is a finite field extension of K containing $K(P)$, and σ is the map $\mathbb{M}_L \to \mathscr{X}$ corresponding to P (cf. [66, Section 1].

Following Vojta [66, Section 1], we define the truncated counting function from higher-dimensional Nevanlinna theory (cf. [65, Section 3.4], [38, Section 2.2]).

Definition 2.2. — Let X be a smooth complete variety over a number field or one-variable function field K, and S be a finite set of places of K containing the archimedean places. For an effective Cartier divisor D on X and a point $P \in X(\overline{K}) \setminus \mathrm{Supp}\, D$, define the *truncated* *counting function* by

$$N^{(1)}_{K,S}(D,P) := \frac{1}{[K(P):K]} \sum_{\substack{\nu\in M_{K(P)}\\ \nu\nmid S}} \min\{1, n_\nu(D)\}\,\mu_\nu$$

Relative to the local height functions, the truncated counting function can be given as

$$N^{(1)}_{K,S}(D,P) := \frac{1}{[K(P):K]} \sum_{\substack{\nu\in M_{K(P)}\\ \nu\nmid S}} \chi_{\mathbb{R}_{>0}}\left(\lambda_{D,\nu}(P)\right)\mu_\nu,$$

where $\chi_{\mathbb{R}_{>0}}$ is the characteristic function for the positive real numbers.
We also recall the definition of a normal crossings divisor (cf. [66, Section 1], [36, Definition 4.1.1.], [62, Tag 0CBN]).

Definition 2.3 **(Normal crossings divisor).** — A Cartier divisor D on a smooth variety X is a *normal crossings divisor* if D is a formal sum $\sum_i D_i$ of distinct irreducible subvarieties D_i and if it can be represented for every point $P \in X$ by a principal divisor $(x_1 \dots x_r)$ in the completed local ring $\widehat{\mathscr{O}}_{P,X}$, where $x_1, \dots, x_r$ is part of a regular sequence for $\widehat{\mathscr{O}}_{P,X}$.

Finally, we define the logarithmic discriminant of a finite field extension. Here we follow Looper [38, Section 2.2], but a uniform definition can also be given by observing that there is a finite morphism

$$(\mathbb{M}_L)_{\mathrm{na}} \to (\mathbb{M}_K)_{\mathrm{na}},$$

and then taking the degree of its ramification divisor (viz. [66, Section 1]).

Definition 2.4 **(Logarithmic discriminant).** — Let L/K be a finite extension of a number field or one-variable function field of characteristic zero, let $D_{L/K}$ be its discriminant, and if L/K is a function field then let $g(L)$ denote its genus. Define the *logarithmic discriminant* of L/K to be

$$d_{L/K} := \begin{cases} \frac{\log\left(\sum_{\nu\in(\mathbb{M}_K)_{\mathrm{na}}} |D_{L/K}|_\nu^{-r_\nu}\right)}{[L:K]} & \text{if } L/K \text{ is a number field,} \\ \frac{2g(L)-2}{[L:K]} & \text{if } L/K \text{ is a function field.} \end{cases}$$

Remark 2.5. — Looper [38] refers to Kim–Thakur–Voloch [31, Section 2] for the definition of the genus of a function field in one variable. It can be defined instead in terms of the genus of $\mathbb{M}_K$ or in terms of divisors and the Riemann–Roch theorem (viz. [10, Section 2.1], [35, Section I.2], or [63, Definition 1.4.15]).

2.2. Vojta's *abcde...* conjecture.

— With the *abc* conjecture, a natural question is to ask whether one might expect a similar inequality between the maximum of an n-tuple of positive coprime integers and their radical. Using a higher-dimensional version of Nevanlinna theory, Vojta [66] posed a higher-dimensional generalization of the *abc* conjecture called the "*abcde...* conjecture".

Conjecture 2.6 **(The *abcde...* conjecture** [66, Conjecture 2.3]**).** — *Let X be a smooth complete variety over a number field or one-variable function field K of characteristic 0, S be a finite set of places of K containing the archimedean places, D be a normal crossings divisor on X, K_X be the canonical divisor on X, $\mathscr{A}$ be a big line bundle on X, r be a positive integer,* *and* $\epsilon > 0$. *Then there exists a proper Zariski-closed subset* $\mathcal{Z} = \mathcal{Z}(K, S, X, D, \mathscr{A}, r, \epsilon) \subsetneq X$ *such that*

$$N^{(1)}(D, P) \geq h_{K_X+D}(P) - \epsilon h_{\mathscr{A}}(P) - d_{K(P)/K} + O(1),$$

for all $P \in X(\overline{K}) \backslash \mathcal{Z}$ *with* $[K(P) : K] \leq r$.

Conjecture 2.6 in this original form is known to be false, but can be salvaged with modification as we will mention later. First, we describe how it generalizes the *abc* conjecture (Conjecture 1.5). A triple (a, b, c) of coprime positive integers such that $a + b = c$ corresponds to a point $P = [a : b : -c]$ on the line $X := \{Z_0 + Z_1 + Z_2 = 0\} \cong \mathbb{P}^1_{\mathbb{Q}}$ in $\mathbb{P}^2_{\mathbb{Q}}$. Let $\mathscr{A} := \mathscr{O}(1)$ and let D be the normal crossings divisor $\{Z_0 = 0\} + \{Z_1 = 0\} + \{Z_2 = 0\}$ (in the coordinates of the ambient $\mathbb{P}^2_{\mathbb{Q}}$). Then $h_{\mathscr{A}}(P) = \log(c)$ and $N^{(1)}(D, P) = \operatorname{rad}(abc)$. Notice that $h_{K_X+D}(P) = \log(c) + O(1)$ since $\omega_X \cong \mathscr{O}(-2)$, D corresponds to three distinct points on X, and $\mathscr{O}(K_X + D) \cong \mathscr{O}(1)$. In this example with $S = \{\infty\}$ and $r = 1$, the inequality of Conjecture 2.6 becomes

$$\begin{aligned}\operatorname{rad}(abc) &\geq \log(c) + O(1) - \epsilon\log(c) - 0 + O(1)\\ &\geq (1 - \epsilon)\log(c) + O(1).\end{aligned}$$

With a different ϵ (such as $\epsilon \mapsto 1 - \frac{1}{1+\epsilon}$),

$$\log(c) \leq (1 + \epsilon)\operatorname{rad}(c) + O(1).$$

Exponentiating yields the desired inequality of the *abc* conjecture, with a constant C not necessarily depending on ϵ.
Repeating the same procedure any n yields an analogous conclusion for generic n-tuples $(z_0, \ldots, z_{n-1})$ of coprime positive integers such that $\sum_{i=0}^{n-2} z_i = z_{n-1}$. The *abcde*... conjecture implies that, outside of a proper Zariski-closed subset of the hyperplane $X := \{\sum_{i=0}^{n-1} Z_i = 0\} \cong \mathbb{P}^{n-1}_{\mathbb{Q}}$, all such tuples satisfy the inequality

$$z_{n-1} \leq C(\epsilon)\operatorname{rad}\left(\prod_{i=0}^{n-2} z_i\right)^{1+\epsilon}. \tag{1}$$

The nomenclature of the "*abcde*... conjecture" for Conjecture 2.6 is evident in this consequence, if one were to relabel $(z_0, \ldots, z_{n-1})$ as $(a, b, c, d, e, \ldots)$.
For $n = 4$, inequality 1 is the *abcd* conjecture described by Granville [21, Section 8] (cf. [20, Chapter IV.3]).

***Conjecture 2.7* (The *abcd* conjecture, version 1** [21, Section 8]**).** — *For every positive real number* ϵ, *there exists a positive real number* $C(\epsilon)$ *such that*

$$d < C(\epsilon)\operatorname{rad}(abcd)^{1+\epsilon},$$

for every quadruple (a, b, c, d) *of coprime positive integers for which* $a + b + c = d$ *outside of a Zariski-closed subset.*

Remark 2.8. — Granville [21, Section 8] describes another variant of Conjecture 2.7, removing the "outside of a Zariski-closed subset" condition with a larger exponent in the inequality:

$$d < C(\epsilon)\operatorname{rad}(abcd)^{3+\epsilon}.$$

This variation of the Conjecture 2.7 is the case $n = 4$ of the n-conjecture of Browkin–Brzezinski [3, Section 2] (also sometimes called the *abcd* conjecture, e.g. [56, Section 4], [67, Section 2]), which asserts a general exponent of $2n - 5 + \epsilon$.

Another version of the *abcd* conjecture is described by Looper [38, Conjecture 2.1], using a more general form of inequality 1 in the language of heights. Define

$$\operatorname{rad}(P) = \operatorname{rad}\left([z_0 : \cdots : z_{n-1}]\right) := \frac{1}{[K(P) : K]} \sum_{\substack{\nu \in (M_K)_{\mathrm{na}} \\ \nu(z_i) \neq \nu(z_j) \text{ for some } i,j}} \mu_\nu.$$

Conjecture 2.9 **(The *abcd* conjecture, version 2** [38, Conjecture 2.1]**).** — *Let K be a number field or a 1-dimensional function field of characteristic 0. Let $n \geq 3$, $[Z_1, \ldots, Z_n]$ be the standard coordinates on $\mathbb{P}_K^{n-1}$, and $\mathcal{H} \subset \mathbb{P}_K^{n-1}$ be the hyperplane given by $\sum_i Z_i = 0$. For any $\epsilon > 0$, there is a proper Zariski-closed subset $\mathcal{Z} = \mathcal{Z}(K, \epsilon, n) \subsetneq \mathcal{H}$ and a constant $C_{K,\mathcal{Z},\epsilon,n}$ such that*

$$h(P) < (1 + \epsilon) \operatorname{rad}(P) + C_{K,\mathcal{Z},\epsilon,n},$$

for all $P \in \mathcal{H} \backslash \mathcal{Z}$.

Remark 2.10. — For the remainder of this article, "*abcd* conjecture" will mean the Version 2 (Conjecture 2.9) as described by Looper [38].

The original form of Vojta's *abcde*... conjecture, given in Conjecture 2.6, was shown to be false by Masser [40]. Looper [38, Section 2.2] notes that it can be replaced by the following weaker conjecture (with $P \in X(K)$ and without the logarithmic discriminant).

Conjecture 2.11 **(The weak *abcde*... conjecture** [39, Conjecture 2.2]**).** — *Let X be a smooth complete variety over a number field or one-variable function field K of characteristic 0, S be a finite set of places of K containing the archimedean places, D be a normal crossings divisor on X, K_X be the canonical divisor on X, $\mathscr{A}$ be a big line bundle on X, and $\epsilon > 0$. Then there exists a proper Zariski-closed subset $\mathcal{Z} = \mathcal{Z}(K, S, X, D, \mathscr{A}, r, \epsilon) \subsetneq X$ such that*

$$N^{(1)}(D, P) \geq h_{K_X + D}(P) - \epsilon h_{\mathscr{A}}(P) + O(1),$$

for all $P \in X(K) \backslash \mathcal{Z}$.

This form is only weaker by ignoring the logarithmic discriminant and requiring that P is K-rational rather than be of bounded degree over K. It still implies the *abc* conjecture and the *abcd* conjecture.

3. Uniform boundedness

3.1. Uniform boundedness of torsion points. — In the first decade of the 20th century, Levi conjectured a classification of torsion groups for elliptic curves over $\mathbb{Q}$ (cf. [57, Section 6]) that was later eventually proved by Mazur [41, 42] and then extended to general number fields by Kamienny [29], Kamienny–Mazur [30], and Merel [43].

Theorem 3.1 **(Torsion theorem** [43]**).** — *Fix a positive integer D. There is a positive integer $C(D)$ such that for all elliptic curves E defined over a number field K of degree D, the number of K-rational torsion points on E is uniformly bounded by $C(D)$:*

$$\#E(K)_{\mathrm{tors}} \leq C(D).$$

Remark 3.2. — In fact, the uniform bound of Theorem 3.1 can be made effective by the results of Parent [55].

The natural generalization of the torsion theorem to abelian varieties is an open problem for dimension greater than 1.

Conjecture 3.3 **(Uniform boundedness conjecture for torsion points of abelian varieties).** — *Fix positive integers g and D. There is a positive integer $C(g, D)$ such that for all abelian varieties A of dimension g defined over a number field K of degree D, the number of K-rational torsion points on A is uniformly bounded by $C(g, D)$:*

$$\#A(K)_{\mathrm{tors}} \leq C(g, D).$$

The general uniform boundedness conjecture for torsion points of abelian varieties is wide open. However, it is sufficient to prove the conjecture for Jacobian varieties due to Cadoret–Tamagawa [5]; they show that Conjecture 3.3 for abelian varieties of dimension g follows from the statement of Conjecture 3.3 for Jacobian varieties of dimension $1 + 6^{8g}(8g-1)!\frac{8g(8g-1)}{2}$. They also demonstrated a version of uniform boundedness for the p-primary part of torsion in families of g-dimensional abelian varieties parametrized by curves [4, Theorem 1.1], [6, Corollary 4.3].
Otherwise, there has been partial progress in specific cases, such as for certain abelian surfaces of CM-type (cf. [59, Corollaries 2-3], [17, Théorème 2]) and for abelian varieties with specific anisotropic reduction (cf. [12, Corollary 2 and Theorem 4]). In a parallel to the development of the *abcd* conjecture, it is known due to Clark–Xarles [12, Section 6] that a higher-dimensional analogue of the weak Szpiro's conjecture implies the uniform boundedness of torsion points for Hilbert–Blumenthal abelian varieties.

3.2. Uniform boundedness of preperiodic points.

— In the simplest dynamical setting, we replace endomorphisms of elliptic curves with endomorphisms of projective space $\mathbb{P}^N$.

Theorem 3.4 **(Northcott's theorem** [51]**).** — *Fix positive integers $N \geq 1$ and $d \geq 2$. For all degree d morphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ defined over a number field K, the number of K-rational preperiodic points is finite.*

A uniform boundedness conjecture for preperiodic points of an endomorphism on projective space was posed by Morton–Silverman [50].

Conjecture 3.5 **(Dynamical uniform boundedness conjecture** [50]**).** — *Fix integers $N \geq 1$, $d \geq 2$, and $D \geq 1$. There is a positive integer $C(N, d, D)$ such that for all degree d morphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ defined over a number field K of degree D, the number of K-rational preperiodic points is uniformly bounded by $C(N, d, D)$:*

$$\#\,\mathrm{PrePer}(f, \mathbb{P}^N_K) \leq C(N, d, D).$$

Fakhruddin [16, Corollary 2.4] showed that the dynamical uniform boundedness conjecture implies the uniform boundedness conjecture for torsion points on abelian varieties.
Assuming the *abcd* conjecture, Looper proved a version of the dynamical uniform boundedness conjecture of Morton–Silverman [50], first for unicritical polynomials [38] and then for general single-variable polynomials [39].

Theorem 3.6 **(**[38, 39, Theorem 1.2]**).** — *Fix an integer $d \geq 2$. Let K be a number field or a 1-dimensional function field of characteristic 0. If the abcd conjecture holds, then there is a positive integer $C(d, K)$ such that for any polynomial $f \in K[z]$ (that is furthermore isotrivial if K is a function field) of degree d, the number of K-rational preperiodic points of f is uniformly bounded by $C(d, K)$:*

$$\#\operatorname{PrePer}(f, \mathbb{P}^1_K) \leq C(d, K).$$

The proof of Theorem 3.6 uses several ingredients. A key ingredient is a global equidistribution statement pieced together from local ν-adic Julia-Fatou equidistribution results obtained from the potential theory of algebraically closed and complete non-Archimedean metrized fields via the Berkovich projective line (for more on this general area, see [2]). Besides the input of height machinery and bounds on heights of preperiodic points of polynomial functions (e.g. from [27, 37]), one of the other main ideas is to show that the prime factors of differences of preperiodic points $z_i - z_j$ are typically in the places of bad reduction when f has many preperiodic points. The *abcd* conjecture (Conjecture 2.9) can then be applied to combinatorial "polygons" constructed from preperiodic points giving points on the projective hyperplanes $\mathcal{H}$.
Following Looper [38] with some case-by-case calculations, Panraksa [54] showed that the usual *abc* conjecture implies that there are no rational non-fixed periodic points for large-degree unicritical polynomials.

Theorem 3.7 **(**[54, Theorem 3]**).** — *If the abc conjecture holds, then the number of rational preperiodic points of f is bounded,*

$$\#\operatorname{PrePer}(f_{d,c}, \mathbb{P}^1_{\mathbb{Q}}) \leq 4,$$

for all unicritical polynomials $f_{d,c}(z) := z^d + c \in \mathbb{Q}[z]$ with d sufficiently large. Furthermore, if $c \neq -1$ then $f_{d,c}$ has no rational periodic points of exact period greater than 1 for d sufficiently large.

The idea of the proof of Theorem 3.7 follows the difference-of-preperiodic-points idea used in [38, 39]. The main observation is that by the *abc* conjecture for the positive integer triple $(|Z_1^d|, |Z_2^d|, |(Z_3 - Z_2)Z^{d-1}|)$, the particular system of equations

$$\begin{aligned} Z_2^d - Z_1^d = (Z_3 - Z_2)Z^{d-1} &\neq 0, \\ \gcd(Z_1, Z_2) &= 1, \\ \max\{|Z_1|, |Z_2|, |Z_3|\} &= Z_3, \\ \max\{|Z_1|, |Z_2|, |Z_3|, |Z|\} &> 1, \end{aligned}$$

has no integral solutions (Z_1, Z_2, Z_3, Z) for sufficiently large d. Periodic points of $f_{d,c}$ generate such systems of equations by expressing elements of an orbit as

$$\left\{ z_1 = \frac{Z_1}{Z}, \ldots, z_n = \frac{Z_N}{Z} \right\},$$

and looking at the differences, e.g. $z_3 - z_2 = z_2^d - z_1^3$.
The statement of Theorem 3.7 about the non-existence of rational periodic points for large enough d can be partially lifted from $\mathbb{Q}$ to quadratic number fields K via the following observation. For a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$, $f^{(j)} := f \circ f^{(j-1)}$ is the j-th iterate of f.

Lemma 3.8 **(**[69, Lemma 3.3]**).** — *Let K be a quadratic number field and $f_{d,c}(z) := z^d + c \in \mathbb{Q}[z]$ be a unicritical polynomial of degree $d > 1$. Let $\{z_0, \dots z_{N-1}\} \subset K$ be an orbit of periodic points of $f_{d,c}$ of odd period $N > 1$. If there is a positive integer $j_i < N$ and a nontrivial $\sigma_i \in \mathrm{Gal}(K/\mathbb{Q})$ for each z_i such that*

$$\sigma_i(z_i) = f_{d,c}^{(j_i)}(z_i),$$

then each z_i is a rational number.

Remark 3.9. — The condition of Lemma 3.8 is called the "Galois–dynamics correspondence" [69, Definition 1.1]. This condition is satisfied, for example, when the dynatomic polynomial $\Phi_N(z) \in \mathbb{Q}[z]$ of $f_{d,c}$ is irreducible [69, Proposition 1.5].

Corollary 3.10. — *Let K be a quadratic number field and N be an odd integer greater than 1. If the abcd conjecture holds, the Galois–dynamics correspondence holds for every period-N orbit of $f_{d,c}$ in K, and $c \neq -1$, then $f_{d,c}$ has no K-rational periodic points of exact period greater than 1 for d sufficiently large.*

Using Theorem 3.7, Panraksa observed that the uniform bound of Theorem 3.7 for $K = \mathbb{Q}$ is actually an absolute constant not dependent on d.

Corollary 3.11 **(**[54, Theorem 4]**).** — *If the abcd conjecture holds, then there is a positive integer C such that for all unicritical polynomials $f_{d,c}(z) := z^d + c \in \mathbb{Q}[z]$ with $d \geq 2$,*

$$\#\,\mathrm{PrePer}(f_{d,c}, \mathbb{P}^1_{\mathbb{Q}}) \leq C.$$

While the uniform bound on the number of preperiodic points of unicritical polynomials in Corollary 3.11 is for the rational numbers, there is an unconditional bound on the possible periods of periodic points for number fields K due to a result of Morton–Silverman [50, Corollary B]. Their result for periodic points of endomorphisms over number fields was then extended to preperiodic points over all global fields by Canci–Paladino [8]. We state the number field version of Canci–Paladino's result for the forward orbit $\mathscr{O}_f(P) = \{f^{(j)}(P) \mid j \in \mathbb{N}\}$ of a preperiodic point P of an endomorphism f.

Theorem 3.12 **(**[8, Theorem 1]**).** — *Let K be a number field of degree D, S be a finite set of places of K, and f be an endomorphism of $\mathbb{P}^1_K$ of degree d defined over K with good reduction outside of S. If $P \in \mathbb{P}^1_K$ is a preperiodic point of f and $N := |\mathscr{O}_f(P)|$, then*

$$N \leq \max\left\{\left(2^{16|S|-8} + 3\right)\left(12\,|S| \log\left(5\,|S|\right)\right)^D, \left(\left(12\,|S| + 24\right) \log\left(5\,|S| + 5\right)\right)^{4D}\right\}.$$

Specializing to periodic points of quadratic rational maps f, Canci [7] used Morton–Silverman's bound to show that quadratic rational maps with good reduction outside a set of places S typically do not have periodic points of large period.

Theorem 3.13 ([7, Theorem 1′]). — *Let K be a number field with ring of integers $\mathscr{O}_K$ and let S be a finite set of places of K. There are only finitely many $\mathrm{PGL}_2(\mathscr{O}_{K,S})$-conjugacy classes of quadratic rational maps defined over K with good reduction outside of S and with a periodic point of minimal period greater than* 3.

For periodic points of unicritical polynomials with coefficients in $\mathscr{O}_K$, Panraksa observed in an unpublished note [53] that Theorem 3.12 implies a uniform bound on possible periods.

Corollary 3.14. — *Let K be a number field of degree D with ring of integers $\mathscr{O}_K$. For the unicritical polynomial $f_{d,c}(z) := x^d + c \in \mathscr{O}_K[z]$ with $d > 1$, there is a positive integer C_D depending only on the degree D such that $f_{d,c}$ has no periodic points of period $N \geq C_D$ in K.*

Proof. — Take S to be the set of archimedean places in K. Since $c \in \mathscr{O}_K$, the unicritical polynomial $f_{d,c}$ has good reduction everywhere outside of S. Furthermore, $|S| = r_K + s_K \leq [K:\mathbb{Q}]$. By Theorem 3.12, $N \leq C_D$ where

$$C_D = (12\,(D+1)\log(5\,(D+1)))^{4[K:\mathbb{Q}]}. \qquad \square$$

3.3. Uniform boundedness of small points. — One of the well-known consequences of Szpiro's conjecture due to Hindry–Silverman [24, p. 420] is Lang's height conjecture about a lower bound for the Néron–Tate (canonical) height.

Conjecture 3.15 **(Lang's height conjecture** [34, p. 92]**).** — *Let K be a number field. There is a positive constant $C(K)$ such that for any elliptic curve E defined over K with minimal discriminant Δ_E, the Néron–Tate height of every non-torsion point $P \in E(K)$ satisfies*

$$\widehat{h}(P) \geq C(K)\log(N_{K/\mathbb{Q}}\Delta_E).$$

Remark 3.16. — If E/K has good reduction everywhere, then $N_{K/\mathbb{Q}}\Delta_E = 1$ and the bound in Lang's height conjecture is trivial. In this case, Hindry–Silverman [25, Corollaire 2] obtained a nontrivial lower bound for the Néron–Tate height of non-torsion points $P \in E(K)$ solely in terms of $D := [K:\mathbb{Q}]$:

$$\widehat{h}(P) \geq \left(10^{18}D^3(\log(D))^2\right)^{-1}.$$

For an endomorphism ϕ of $\mathbb{P}^N$ of degree $d \geq 2$, the canonical height $\widehat{h}_\phi$ is the unique real-valued function on $\mathbb{P}^N(\overline{\mathbb{Q}})$ such that $\widehat{h}_\phi(P) = h(P) + O(1)$ and $\widehat{h}_\phi(\phi(P)) = d\widehat{h}_\phi(P)$, where $h(P)$ is the Weil height. It is similarly characterized by the property that the height $\widehat{h}_\phi(P)$ vanishes if and only if P is a preperiodic point for ϕ [61, Theorem 3.22]. Silverman [61, Conjecture 4.98] formulated a dynamical version of Lang's conjecture (not to be confused with the dynamical Mordell–Lang conjecture) that captures how the canonical height $\widehat{h}_f$ describes the "non-preperiodicity" of a point P relative to a dynamical system.

Let Rat_d be the moduli space of rational maps of degree d. The quotient variety $\mathscr{M}_d := \mathrm{Rat}_d / \mathrm{PGL}_2$ by the PGL_2 conjugacy action is a moduli space classifying degree d dynamical systems on $\mathbb{P}^1$ (cf. [61, Section 4.4]). Fix a projective embedding $\mathscr{M}_d \hookrightarrow \mathbb{P}^n$ so that a rational function $\phi \in K(z)$ has a corresponding Weil height from $\langle\phi\rangle \in \mathscr{M}_d(K)$. Let the minimal resultant of ϕ be

$$\mathfrak{R}_\phi := \prod_{\mathfrak{p}} \mathfrak{p}^{\epsilon_{\mathfrak{p}}(\phi)},$$

where $\epsilon_{\mathfrak{p}}(\phi)$ is the greatest exponent of $\mathfrak{p}$ dividing the resultant of the PGL_2-conjugate of ϕ with the best reduction at $\mathfrak{p}$ (cf. [61, Section 4.11]). This allows for the accounting of twists of ϕ which have the same $\mathscr{M}_d$ class but do not necessarily have the same Weil height.

Conjecture 3.17 **(Dynamical Lang conjecture** [61, Conjecture 4.98]**).** — *Fix an embedding of the moduli space $\mathscr{M}_d$ in projective space and let $h_{\mathscr{M}_d}$ denote the associated height function. Let K be a number field and $d \geq 2$ be an integer. Then there is a positive constant $C(K,d)$ such that*

$$\widehat{h}_\phi(P) \geq C(K,d)\max\Big\{\log(N_{K/\mathbb{Q}}\mathfrak{R}_\phi), h_{\mathscr{M}_d}(\langle\phi\rangle)\Big\}.$$

for all rational maps $\phi \in K(z)$ of degree d and all non-preperiodic points $P \in \mathbb{P}^1_K$.

Using the *abcd* conjecture along the same lines as the proof of Theorem 3.6, Looper gives a weaker version of the dynamical Lang conjecture. Here, this version uses the critical height, which is the sum of canonical heights at critical points

$$\widehat{h}_{\mathrm{crit}}(\phi) := \sum_{P \in \mathrm{Crit}(\phi)} \widehat{h}_\phi(P),$$

and is commensurate with the Weil height $h_{\mathscr{M}_d}$ [28, Theorem 1].

Theorem 3.18 **(**[39, Theorem 1.3]**).** — *Fix an integer $d \geq 2$. Let K be a number field or a* 1*-dimensional function field of characteristic* 0*. If the abcd conjecture holds, then there is a constant $C(d,K) > 0$ such that for any polynomial $f \in K[z]$ of degree d and for all $P \in K$, either $\widehat{h}_f(P) = 0$ or*

$$\widehat{h}_f(P) \geq C(d,K)\max\Big\{1, \widehat{h}_{\mathrm{crit}}(f)\Big\}.$$

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