ITERATED INTEGRALS, DIAGONAL CYCLES AND RATIONAL POINTS ON ELLIPTIC CURVES

by

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Abstract. — The theme of this article is the connection between the pro-unipotent fundamental group $\pi_1(X;o)$ of a pointed algebraic curve X, algebraic cycles, iterated integrals, and special values of L-functions. The extension of mixed Hodge structures arising in the second stage in the lower central series of $\pi_1(X;o)$ gives rise to a supply of complex points on the Jacobian $\mathrm{Jac}(X)$ of X indexed by Hodge cycles on $X \times X$. The main results of this note relate these points to the Abel-Jacobi image of Gross-Kudla-Schoen's modified diagonal in X^3, and express this Abel-Jacobi image in terms of iterated integrals. The resulting formula is the basis for the practical complex-analytic calculations of these points when X is a modular (or Shimura) curve, a setting where the recent work **[YZZ]** of X. Yuan, S. Zhang and W. Zhang relates their non-triviality to special values of certain L-series attached to modular forms.

Résumé. — Le thème de cet article est le groupe fondamental pro-unipotent $\pi_1(X;o)$ d'une courbe algébrique X munie d'un point base et ses relations avec les cycles algébriques, les intégrales itérées, et les valeurs spéciales des fonctions L. Les extensions de structures de Hodge mixtes qui apparaissent dans le quotient de $\pi_1(X,o)$ par le deuxième terme de sa série centrale descendante donnent lieu à des points complexes de la Jacobienne $\mathrm{Jac}(X)$ de X. Ces points sont paramétrés par les cycles de Hodge de $X \times X$. On les relie aux images de la diagonale modifiée de Gross-Kudla-Schoen dans X^3 par l'application d'Abel-Jacobi, en exprimant ces images au moyen d'intégrales itérées. La formule explicite qui s'en dégage a permis le calcul numérique de ces points lorsque X est une courbe modulaire (ou une courbe de Shimura), contexte dans lequel les travaux récents **[YZZ]** de X. Yuan, S. Zhang et W. Zhang les relient aux valeurs spéciales de certaines fonctions L attachées à des formes modulaires.

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Introduction

Let (Y, o) be a pointed algebraic variety over the complex numbers. Its pro-unipotent fundamental group $\pi_1(Y; o)$ is a rich source of links between the topology, geometry and arithmetic of the variety: see for example the works of Bloch [**Bl77**], Chen [**Chen**], Deligne [**De**], Drinfeld [**Dr**], Hain [**Hain3**] and Kim [**Ki**] and the references therein.
Among these developments, the contributions of Carlson, Clemens and Morgan [**CCM**], Harris [**Har**] and Pulte [**Pu**] in the eighties exhibited examples where the values of Chen's iterated integrals –which provide an explicit description of the underlying mixed Hodge structure of the de Rham realization of $\pi_1(Y; o)$– coincide with the Abel-Jacobi images of a suitable null-homologous cycle on Y or on some other variety that one may naturally construct out of Y. In geometric terms, these results can be phrased as the equality of two points in an intermediate jacobian of the variety: one, constructed as the class of the periods afforded by certain iterated integrals, and the other obtained as the image of a suitable cycle under the Abel-Jacobi map.
Let $F \subset \mathbb{C}$ be a field embedded in the field of complex numbers, X a smooth projective curve over F and $Y = X \setminus \{\infty\}$ the complement of a single rational point $\infty \in X(F)$. Choose a base point $o \in Y(F)$. In this article we focus on the construction of a supply of complex points

$$\kappa_{\mathrm{dR}}(\xi) \in \mathrm{Jac}(X)(\mathbb{C}) \tag{1}$$

on the jacobian of X, indexed by the set of Hodge classes

$$\mathrm{Hdg}(H^1(X) \otimes H^1(X)) := \{\xi : \mathbb{Z}(-1) \to H^1(X) \otimes H^1(X)\}$$

in the Hodge structure underlying $H^1(X) \otimes H^1(X)$. This construction has also been extensively studied by Pulte [**Pu**] and Kaenders [**Ka**] and we review it in §1. These points arise from the pro-unipotent fundamental group $\pi_1(Y(\mathbb{C}); o)$ and therefore depend on the choice of o, and their construction exploits the canonical isomorphism

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-1), H^1_B(X)) \simeq \frac{\Omega^1(X)^\vee}{H_1(X(\mathbb{C}), \mathbb{Z})} \simeq \mathrm{Jac}(X)(\mathbb{C}) \tag{2}$$

between the module of extensions of $\mathbb{Z}(-1)$ by $H^1_{\mathrm{B}}(X)$ in the category of mixed Hodge structures, and the set of complex points of $\mathrm{Jac}(X)$, as described e.g. in [**Ca**].
Assume now that $\xi = \xi_Z$ is the Hodge class associated by the cycle class map with an algebraic cycle $Z \in \mathrm{Pic}(X \times X)$, as detailed in (44). Section 2 associates to each divisor Z in $X \times X$ an algebraic correspondence $\Pi_Z : X^3 \dashrightarrow X$ between X^3 and X and our first main theorem is:

Theorem 1. — *For every divisor class $Z \in \mathrm{Pic}(X^2)$,*

$$\kappa_{\mathrm{dR}}(\xi_Z) = \Pi_Z(\Delta_{\mathrm{GKS}}) \in \mathrm{Jac}(X)(\mathbb{C}), \tag{3}$$

where
- $\Pi_Z : \mathrm{CH}^2(X^3)_0 \longrightarrow \mathrm{CH}^1(X)_0 = \mathrm{Jac}(X)$ *is the transformation induced by* Π_Z *by functoriality, and*
- $\Delta_{\mathrm{GKS}} \in \mathrm{CH}^2(X^3)_0$ *is the null-homologous cycle of codimension* 2 *in* X^3 *with base-point* o *studied by Gross, Kudla and Schoen* [**GrKu**], [**GrSc**], *cf.* (47).

This theorem spells out a relationship between diagonal cycles and the pro-unipotent fundamental group which is hinted at in the penultimate sentence of §5.1 of [**Zh2**]; the reader is encouraged to consult §5 of loc.cit. for further discussion of Gross-Kudla-Schoen cycles and their arithmetic applications. Since the right-hand side of (3) is purely algebraic, we derive the following corollary:

Corollary 2. — *If Z belongs to $\mathrm{Pic}(X\times X)(F)$ then $\kappa_{\mathrm{dR}}(\xi_Z)$ is an F-rational point of $\mathrm{Jac}(X)$.*

The proof of Theorem 1 is complex analytic: it proceeds by relating–in §1 and §2, respectively– the left and right-hand side of equality (3) to Chen's iterated integrals. The resulting explicit formula (Theorem 2.5) was one of the main motivations for this note and underlies the numerical calculations of Chow-Heegner points that are carried out in [**DDLR**] when X is a modular curve. The authors also hope that Theorem 1 could form the basis for an eventual approach to the "Stark-Heegner points" of [**DP**] via iterated integrals.

Beyond the general expectation alluded to above that the pro-unipotent fundamental group $\pi_1(Y;o)$ of a variety should encode the Abel-Jacobi images of non-trivial null-homologous algebraic cycles, it is also natural to expect, on the grounds of the conjectures of Bloch-Kato [**BK**] linking algebraic cycles to L-functions, that a good deal of the non-abelian arithmetic of $\pi_1(Y;o)$ should be accounted for by the leading terms of the L-function of the motives appearing as Jordan-Hölder constituents of $\pi_1(Y;o)$. There are however few examples in the literature in which this principle is illustrated [(1)]. A celebrated instance is afforded by the curve $Y=\mathbb{P}_1-\{0,1,\infty\}$, whose pro-unipotent fundamental group is an iterated extension of Tate motives. It is shown in [**De**] that the extension classes that arise in it encode special values of the Riemann zeta function at certain (non-critical) points. Theorem 3.7 in §3 describes analogous phenomena when Y is a punctured modular curve attached to a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, or a Shimura curve, exploiting [**YZZ**].

Let us close this introduction by relating Theorem 1 to existing results in the literature. In our notations and the normalizations adopted in definition (43) of the points $\kappa_{\mathrm{dR}}(\xi)$ –which are tailored so that there is no dependence on the choice of the point ∞–, the main theorem of Kaenders [**Ka**] says that when $Z=X_{12}\in\mathrm{Pic}(X\times X)$ is the class of the curve X diagonally embedded in $X\times X$, we have

$$\kappa_{\mathrm{dR}}(\xi_{X_{12}})=(2g-2)o-K\in\mathrm{Jac}(X), \tag{4}$$

where g stands for the genus of X and K is any canonical divisor on the curve. Note that, since K is an F-rational divisor, the conclusion of Corollary 2 follows directly from (4) for this particular choice of Z.

Kaenders proves (4) by showing that a suitable translate of $\kappa_{\mathrm{dR}}(\xi_{X_{12}})$ is equal to twice Riemann's constant; his result then follows by an immediate application of the Riemann-Roch theorem. It can also be recovered directly from Theorem 1: see Corollary 2.8.

It is also interesting to compare Theorem 1 with the work of B. Harris [**Har**], as recast by Pulte in [**Pu**]. For any $n\in\mathbb{Z}$, let $[n]\in\mathrm{End}(\mathrm{Jac}(X))$ denote the endomorphism given by

1. while the connection between algebraic cycles and special values of L-functions as predicted by [**BK**] has already borne some fruit: cf. e.g. [**GrZa**], [**Ne**], [**Zh**], [**YZZ**].

multiplication-by-n. Define

$$X_o := \{[P-o] : P \in X\}, \quad X_o^- := [-1]_* X_o = \{[o-P] : P \in X\} \subset \mathrm{Pic}^0(X)$$

and let $C_o := X_o - X_o^- \in \mathrm{CH}^{g-1}(\mathrm{Jac}(X))_0$ be *Ceresa's cycle*, a null-homologous cycle of codimension $g-1$ on the jacobian of X.
Every choice of Hodge class $\xi \in \mathrm{Hdg}(H^1_B(X)^{\otimes 2})$ singles out a natural map [2]

$$\pi_\xi : \mathrm{CH}^{g-1}(\mathrm{Jac}(X))_0(\mathbb{C}) \longrightarrow \mathrm{Jac}(X)(\mathbb{C}), \tag{5}$$

and Theorem 4.9 of **[Pu]** asserts that

$$\kappa_{\mathrm{dR}}(\xi) = 2\pi_\xi(C_o) \in \mathrm{Jac}(X)(\mathbb{C}). \tag{6}$$

Notice that Corollary 2 does not follow automatically from (6), as it is not a priori obvious that the map π_ξ is Galois equivariant.
Our Theorem 1 is in the same spirit of Harris-Pulte's, in that both relate the points $\kappa_{\mathrm{dR}}(\xi)$ arising from $\pi_1(Y(\mathbb{C}); o)$ to algebraic cycles on some algebraic variety intimately related to X, namely X^3 and $\mathrm{Jac}(X)$, respectively. These two varieties and their relevant Chow groups are related through their intermediate jacobians by the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^2(X^3)_0(\mathbb{C}) & \longrightarrow & J^2(X^3) = \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3,\mathbb{C})/H_3(X^3,\mathbb{Z}) \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mathrm{AJ}(\mu_*)} \\ \mathrm{CH}^{g-1}(\mathrm{Jac}(X))_0(\mathbb{C}) & \longrightarrow & J_1(\mathrm{Jac}(X)) = \mathrm{Fil}^2 H^3_{\mathrm{dR}}(\mathrm{Jac}(X),\mathbb{C})/H_3(\mathrm{Jac}(X),\mathbb{Z}), \end{array} \tag{7}$$

where the horizontal arrows are the Abel-Jacobi maps AJ_{X^3} on X^3 and $\mathrm{AJ}_{\mathrm{Jac}(X)}$ on $\mathrm{Jac}(X)$, respectively, the left-most vertical arrow is the transformation induced by functoriality by the morphism $\mu : X^3 \to \mathrm{Jac}(X)$ given by the rule $(x_1,x_2,x_3) \mapsto x_1+x_2+x_3-3o$, and the vertical arrow $\mathrm{AJ}(\mu)$ is the natural surjection given by the Künneth decomposition of $H^3(X^3)$ and the classical isomorphism $H^3(\mathrm{Jac}(X)) = \Lambda^3 H^1(X)$.
It readily follows from the definitions that

$$\mu_*(\Delta_{\mathrm{GKS}}) = [3]_* X_o - 3[2]_* X_o + 3X_o \in \mathrm{CH}^{g-1}(\mathrm{Jac}(X))_0$$

and Colombo and van Geemen proved in **[CG**, Proposition 2.9**]** that

$$\mathrm{AJ}_{\mathrm{Jac}(X)}(\mu_*(\Delta_{\mathrm{GKS}})) = 3\mathrm{AJ}_{\mathrm{Jac}(X)}(C_o). \tag{8}$$

This result lies far from proving Theorem 1 via (6), as the maps μ_* and $\mathrm{AJ}(\mu_*)$ have non-trivial kernel. In fact, the combination of Theorem 1 with Pulte's (6) yields the equality

$$\Pi_Z(\Delta_{\mathrm{GKS}}) = 2\pi_{\xi_Z}(C_o) \in \mathrm{Jac}(X)(\mathbb{C}) \quad \text{for all } Z \in \mathrm{Pic}(X \times X), \tag{9}$$

which is yet another set of non-trivial relationships between Gross-Kudla-Schoen's cycle $\Delta_{\mathrm{GKS}} \in \mathrm{CH}^2(X^3)_0$ and Ceresa's $C_o \in \mathrm{CH}^{g-1}(\mathrm{Jac}(X))_0$.

2. In the notation of §1.2, π_ξ is the composition of the Abel-Jacobi map $\mathrm{CH}^{g-1}(\mathrm{Jac}(X))_0(\mathbb{C}) \stackrel{\mathrm{AJ}}{\to} J_1(\mathrm{Jac}(X))$, the monomorphism $J_1(\mathrm{Jac}(X)) \hookrightarrow \mathrm{Ext}^1_{\mathrm{MHS}}(H^1_B(X)^{\otimes 2}, H^1_B(X))$ considered by Pulte in **[Pu]**, the projection $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1_B(X)^{\otimes 2}, H^1_B(X)) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-1), H^1_B(X))$ given by ξ as in (36) and the isomorphism (2).

1. Fundamental groups and mixed motives

Let F be a number field equipped with an embedding $F \hookrightarrow \mathbb{C}$ into the field of complex numbers, and write $\bar{F}$ for the Galois closure of F in $\mathbb{C}$; as before, recall that $G_F := \mathrm{Gal}\,(\bar{F}/F)$. Let X be a smooth, complete algebraic curve of genus $g \geq 2$ over F, and let $Y = X \setminus S$ for some finite, G_F-stable set S of points on $X(\bar{F})$. Fix a base point $o \in Y(F)$; the integral group ring $\mathbb{Z}[\Gamma]$ of the fundamental group $\Gamma := \pi_1(Y(\mathbb{C}); o)$ of the Riemann surface underlying

$$Y_{\mathbb{C}} = Y \times_{\mathrm{Spec}(F)} \mathrm{Spec}(\mathbb{C})$$

is equipped with a decreasing filtration

$$\mathbb{Z}[\Gamma] \supset I \supset I^2 \supset \cdots \supset I^j \supset \cdots$$

by powers of the augmentation ideal.
The first few successive quotients in this filtration are given by

$$\mathbb{Z}[\Gamma]/I = \mathbb{Z}; \qquad \sum m_\sigma \sigma \mapsto \sum m_\sigma; \tag{10}$$

$$I/I^2 = \Gamma_{\mathrm{ab}} := \Gamma/[\Gamma,\Gamma]; \qquad \sum m_\sigma \sigma \mapsto \prod \sigma^{m_\sigma}; \tag{11}$$

$$I^2/I^3 = (\Gamma_{\mathrm{ab}} \otimes \Gamma_{\mathrm{ab}})/\mathcal{K}, \tag{12}$$

where in the last identification, $\mathcal{K}$ is the kernel of the natural surjective homomorphism

$$j : \Gamma_{\mathrm{ab}} \otimes \Gamma_{\mathrm{ab}} \longrightarrow I^2/I^3, \qquad \gamma_1 \otimes \gamma_2 \mapsto (\gamma_1 - 1)(\gamma_2 - 1). \tag{13}$$

Let $M^\vee := \hom(M, \mathbb{Z})$ be the $\mathbb{Z}$-linear dual of a $\mathbb{Z}$-module M. Formula (11) implies that

$$(I/I^2)^\vee = \hom(\Gamma, \mathbb{Z}) = H^1_{\mathrm{B}}(Y), \tag{14}$$

where $H^1_{\mathrm{B}}(Y) = H^1(Y(\mathbb{C}), \mathbb{Z})$ refers to the singular Betti cohomology of the smooth real manifold underlying $Y(\mathbb{C})$. Furthermore, it can be shown that the map

$$j^\vee : (I^2/I^3)^\vee \hookrightarrow H^1_{\mathrm{B}}(Y)^{\otimes 2}$$

identifies $(I^2/I^3)^\vee$ with the kernel of the map

$$H^1_{\mathrm{B}}(Y)^{\otimes 2} \longrightarrow H^2_{\mathrm{B}}(Y)$$

given by the cup product. (Cf. [**Hain3**, Cor. 8.2] for more details.) Letting $H^1_{\mathrm{B}}(Y)^{\otimes 2}_{\cup=0}$ denote this kernel, the map $j^\vee$ induces an identification

$$(I^2/I^3)^\vee = H^1_{\mathrm{B}}(Y)^{\otimes 2}_{\cup=0}. \tag{15}$$

By dualising the exact sequence

$$0 \longrightarrow I^2/I^3 \longrightarrow I/I^3 \longrightarrow I/I^2 \longrightarrow 0,$$

invoking (14) and (15), and setting

$$M_{\mathrm{B}} := (I/I^3)^\vee,$$

we obtain the fundamental exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow H^1_{\mathrm{B}}(Y) \longrightarrow M_{\mathrm{B}} \longrightarrow H^1_{\mathrm{B}}(Y)^{\otimes 2}_{\cup=0} \longrightarrow 0. \tag{16}$$

Note that $H^1_{\mathrm{B}}(Y)^{\otimes 2}_{\cup=0} = H^1_{\mathrm{B}}(Y)^{\otimes 2}$ as soon as $S \neq \emptyset$, because in this case $H^2_{\mathrm{B}}(Y) = \{0\}$. The abelian groups that appear on the left and right of (16) are not just "unadorned" $\mathbb{Z}$-modules, but the *Betti realisations* of certain motives over F. In the terminology of [**De**], the exact

sequence (16) suggests that M_B ought to be interpreted as the Betti realization of a putative mixed motive $M^?$ (in the sense of **[De]**, say) which would arise as an extension in that category of the motive $H^1(Y)^{\otimes 2}_{\cup=0}$ (which is pure of weight 2 when $Y = X$) by the motive $H^1(Y)$ (which is pure of weight 1 when $Y = X$). Implicit in the existence of $M^?$, which we do not address in this paper, are the following concrete consequences:

1. The existence of a decreasing, separated and exhaustive *Hodge filtration* on
$$M_{\mathrm{dR},\mathbb{C}} := M_{\mathrm{B}} \otimes \mathbb{C},$$
for which the complexification
$$0 \longrightarrow H^1_{\mathrm{dR}}(Y/\mathbb{C}) \xrightarrow{i} M_{\mathrm{dR},\mathbb{C}} \xrightarrow{\mathrm{p}} H^1_{\mathrm{dR}}(Y/\mathbb{C})^{\otimes 2}_{\cup=0} \longrightarrow 0 \tag{17}$$
of (16) becomes an exact sequence of filtered complex vector spaces.
2. The existence, for each rational prime p, of the p-adic étale realisation $M_{\mathrm{et},p}$, fitting into an exact sequence of continuous $\mathbb{Q}_p[G_F]$-modules
$$0 \longrightarrow H^1_{\mathrm{et}}(X_{\bar{F}}, \mathbb{Q}_p) \longrightarrow M_{\mathrm{et},p} \longrightarrow H^1_{\mathrm{et}}(X_{\bar{F}}, \mathbb{Q}_p)^{\otimes 2}_{\cup=0} \longrightarrow 0 \tag{18}$$
which is the p-adic étale counterpart of (16).

There are other aspects as well (notably, the cristalline realisations attached to the fundamental groups of the reduction of Y modulo p, equipped with its semi-linear frobenius endomorphism at p and a comparison isomorphism with the de Rham cohomology over p-adic fields) which shall not be touched upon at all here. In this note we focus almost exclusively on (17).

Let us briefly recall the well-known mixed Hodge structures underlying the Betti cohomology group $H^1_{\mathrm{B}}(Y)$ and its tensor square. Its weight filtration arises from the exact sequence
$$0 \longrightarrow H^1_{\mathrm{B}}(X) \longrightarrow H^1_{\mathrm{B}}(Y) \longrightarrow \oplus_{P\in S}\mathbb{Z}(-1) \longrightarrow \mathbb{Z}(-1) \longrightarrow 0, \tag{19}$$
and is given by
$$W_j H^1_{\mathrm{B}}(X) = \begin{cases} 0 & \text{if } j \leq 0, \\ H^1_{\mathrm{B}}(X) & \text{if } j = 1, \\ H^1_{\mathrm{B}}(Y) & \text{if } j \geq 2. \end{cases} \tag{20}$$
The *de Rham cohomology* $H^1_{\mathrm{dR}}(Y(\mathbb{C}))$ is the cohomology of the de Rham complex of smooth $\mathbb{C}$-valued differential forms on $Y(\mathbb{C})$ with at worst logarithmic poles at the points in S. There is a canonical identification $H^1_{\mathrm{B}}(Y) \otimes \mathbb{C} = H^1_{\mathrm{dR}}(Y(\mathbb{C}))$ arising from integration of smooth differential forms against smooth chains, and its Hodge filtration is given by
$$\mathrm{Fil}^j H^1_{\mathrm{dR}}(Y(\mathbb{C})) = \begin{cases} H^1_{\mathrm{dR}}(Y(\mathbb{C})) & \text{if } j \leq 0, \\ \Omega^1_{\log}(Y(\mathbb{C})) & \text{if } j = 1, \\ 0 & \text{if } j \geq 2, \end{cases} \tag{21}$$
where
$$\Omega^1_{\log}(Y(\mathbb{C})) = \{\omega \in \Omega^1(Y(\mathbb{C})) \quad \text{with} \quad \mathrm{ord}_P(\omega) \geq -1, \quad \forall P \in S\}$$
and $\mathrm{ord}_{\mathrm{P}}(\omega)$ denotes the valuation of the differential ω at the point $P \in X(\mathbb{C})$.

Because the Riemann surface $Y(\mathbb{C})$ arises from an algebraic curve over F, we can identify $H^1_{\mathrm{dR}}(Y(\mathbb{C}))$ with $H^1_{\mathrm{dR}}(Y/F) \otimes \mathbb{C}$, where
$$H^1_{\mathrm{dR}}(Y/F) := \mathbb{H}^1(0 \to \mathcal{O}_Y \to \Omega^1(Y) \to 0) \tag{22}$$

is the *algebraic de Rham cohomology* of Y/F, defined as the hypercohomology of the de Rham complex of sheaves of regular differential forms on Y. The Hodge filtration on $H^1_{\mathrm{dR}}(Y/F)$ may also be described concretely by

$$\mathrm{Fil}^j H^1_{\mathrm{dR}}(Y/F) = \begin{cases} H^1_{\mathrm{dR}}(Y/F) \simeq \Omega^{II}(Y)/dF(Y) & \text{if } j \leq 0, \\ \Omega^1_{\log}(Y/F) & \text{if } j = 1, \\ 0 & \text{if } j \geq 2, \end{cases} \tag{23}$$

where

1. $\Omega^{II}(Y)$ is the space of rational differential forms on Y over F with vanishing residues at all points of Y; these are also called *differentials of the second kind* on Y;
2. $F(Y)$ is the field of rational functions on Y over F;
3. $\Omega^1_{\log}(Y/F)$ is the subspace of $\Omega^1(Y/F)$ consisting of differentials ω satisfying $\mathrm{ord}_P(\omega) \geq -1$ for all $P \in S$.

The tensor product $H^1_{\mathrm{B}}(Y)^{\otimes 2}$ is again a mixed Hodge structure, and the Hodge filtration on $H^1_{\mathrm{dR}}(Y/F)^{\otimes 2}$ is given by

$$\mathrm{Fil}^j H^1_{\mathrm{dR}}(Y/F)^{\otimes 2} = \begin{cases} H^1_{\mathrm{dR}}(Y/F)^{\otimes 2} & \text{if } j \leq 0; \\ \Omega^1_{\log}(Y/F) \otimes H^1_{\mathrm{dR}}(Y/F) + H^1_{\mathrm{dR}}(Y/F) \otimes \Omega^1_{\log}(Y/F) & \text{if } j = 1; \\ \Omega^1_{\log}(Y/F)^{\otimes 2} & \text{if } j = 2; \\ 0 & \text{if } j \geq 3. \end{cases}$$

1.1. Iterated integrals and the Hodge filtration on $M_{\mathrm{dR},\mathbb{C}}$. — The extension M_B of $H^1_{\mathrm{B}}(Y)^{\otimes 2}$ by $H^1_{\mathrm{B}}(Y)$ arising in (16) is equipped with a natural mixed Hodge structure. The definition of the Hodge filtration on $M_{\mathrm{dR},\mathbb{C}}$ was given by R. Hain in **[Hain2]**, and rests on the explicit description of $M_{\mathrm{dR},\mathbb{C}}$ in terms of K.-T. Chen's iterated integrals of smooth differential forms on path spaces. We now briefly recall this description, referring the reader to **[Chen]**, **[Hain3]** and the references therein for further details.
The *path space* on Y based at o, denoted $\mathbb{P}(Y; o)$, is the set of piecewise smooth paths

$$\gamma : [0, 1] \longrightarrow Y(\mathbb{C}), \qquad \text{with } \gamma(0) = o.$$

The universal covering space of $Y(\mathbb{C})$, denoted $\tilde{Y}$, is the space of homotopy classes in $\mathbb{P}(Y; o)$. It is conformally equivalent to the open unit disc in $\mathbb{C}$. The group $\Gamma = \pi_1(Y(\mathbb{C}); o)$ acts on it transitively and without fixed points, and the map $\gamma \mapsto \gamma(1)$ identifies the quotient $\tilde{Y}/\Gamma$ with $Y(\mathbb{C})$. If η is a smooth closed one-form (resp. a meromorphic one-form of the second kind) on $Y(\mathbb{C})$, then it admits a smooth (resp. meromorphic) primitive F_η on $\tilde{Y}$, defined by the rule

$$F_\eta(\gamma) := \int_0^1 \gamma^* \eta.$$

The *basic iterated integral* attached to smooth differential forms $\omega_1, \ldots, \omega_n$ along a path $\gamma \in \mathbb{P}(Y; o)$ is defined to be

$$\int_\gamma \omega_1 \cdot \omega_2 \cdots \omega_n := \int_{0 \leq t_n \leq t_{n-1} \leq \cdots \leq t_1 \leq 1} \gamma^*(\omega_1)(t_1) \cdots \gamma^*(\omega_n)(t_n). \tag{24}$$

The integer n is called the *length* of this basic iterated integral. In particular, when $n = 2$, the basic iterated integral attached to η_1 and η_2 is given by the formula

$$\int_\gamma \eta_1 \cdot \eta_2 := \int_\gamma \eta_1 F_{\eta_2} = \int_0^1 \gamma^*(\eta_1 F_{\eta_2}).$$

It can also be expressed in terms of $\eta_1 \wedge \eta_2$, *viewed as a two-form on* $X \times X$, as the expression

$$\int_\gamma \eta_1 \cdot \eta_2 := \int_{T_\gamma} \eta_1 \wedge \eta_2, \qquad \text{where } T_\gamma = \{(\gamma(r), \gamma(s)), \quad 0 \leq s \leq r \leq 1\} \subset X \times X. \tag{25}$$

An *iterated integral* is any linear combination of basic iterated integrals, of possibly differing lengths, viewed as a function on $\mathbb{P}(Y; o)$. The length of an iterated integral is then defined to be the maximal length arising in its expression as a linear combination of basic iterated integrals.
An iterated integral is said to be *homotopy invariant* if its value on a path γ depends only on the homotopy class of γ. The space $\mathrm{II}(Y)$ of homotopy invariant iterated integrals will be viewed as a subspace of the space of $\mathbb{C}$-valued functions on Γ, in the obvious way. By extending $J \in \mathrm{II}(Y)$ to the group ring $\mathbb{C}[\Gamma]$ by $\mathbb{C}$-linearity, we will exploit the resulting natural inclusion

$$\mathrm{II}(Y) \subset \mathbb{C}[\Gamma]^\vee$$

to realise $\mathrm{II}(Y)$ as a space of complex functionals on $\mathbb{C}[\Gamma]$. For each n, let $\mathrm{II}^{\leq n}(Y)$ denote the subspace of homotopy invariant iterated integrals of length $\leq n$. This space plays an important role in our description of $M_{\mathrm{dR},\mathbb{C}}$, because of the following facts:

1. Any element $J \in \mathrm{II}^{\leq n}(Y) \subset \mathbb{C}[\Gamma]^\vee$ vanishes on I^{n+1}, and hence gives rise to a well-defined element of $(I/I^{n+1})^\vee \otimes_{\mathbb{Z}} \mathbb{C}$.
2. The natural map

$$\mathrm{II}^{\leq n} \longrightarrow (I/I^{n+1})^\vee \otimes_{\mathbb{Z}} \mathbb{C} \tag{26}$$

is an isomorphism. In particular,

$$M_{\mathrm{dR},\mathbb{C}} = \mathrm{II}^{\leq 2}(Y). \tag{27}$$

The following lemma is helpful in describing $\mathrm{II}^{\leq 2}(Y)$ concretely.

Lemma 1.1. — *Let η_1 and η_2 be smooth closed one-forms on $Y(\mathbb{C})$ whose classes in de Rham cohomology are orthogonal with respect to the Poincaré pairing, and let α be a smooth one-form on $Y(\mathbb{C})$ satisfying the differential equation*

$$d\alpha = \eta_1 \wedge \eta_2. \tag{28}$$

1. *The iterated integral*

$$J_{\eta_1 \otimes \eta_2, \alpha}(\gamma) := \int_\gamma (\eta_1 \cdot \eta_2 - \alpha)$$

is homotopy invariant.

2. *The image of $J_{\eta_1 \otimes \eta_2, \alpha} \in M_{\mathrm{dR},\mathbb{C}}$ under the projection* p *of* (17) *is the element $[\eta_1] \otimes [\eta_2]$.*
3. *The space $\mathrm{II}^{\leq 2}(Y)$ is spanned by iterated integrals of the form $J_{\eta_1 \otimes \eta_2, \alpha}$.*

Proof. — The first claim follows directly from the fact that the differential one-form on $\tilde{Y}$ appearing in the expression

$$J_{\eta_1\otimes\eta_2,\alpha}(\gamma) = \int_\gamma \eta_1 F_{\eta_2} - \alpha$$

is closed. The second statement follows from a direct calculation showing that

$$J_{\eta_1\otimes\eta_2,\alpha}((\gamma_1-1)(\gamma_2-1)) = \int_{\gamma_1}\eta_1\int_{\gamma_2}\eta_2.$$

Finally, given any $J \in M_{\mathrm{dR},\mathbb{C}}$, after writing $\mathrm{p}(J) = \sum \omega_i \otimes \eta_i$, it can be checked that

$$J = \left(\sum J_{\omega_i\otimes\eta_i,\alpha_i}\right) - J_{0,0,\alpha}$$

for suitable smooth differentials α_i on $Y(\mathbb{C})$ and $\alpha \in H^1_{\mathrm{dR}}(Y(\mathbb{C}))$. □

In the special case where $Y(\mathbb{C}) = X(\mathbb{C}) - \{\infty\}$ with X projective, the smooth differential α on $Y(\mathbb{C})$ can be expressed in terms of the Green's function $g_{\eta_1\wedge\eta_2,\infty}$ attached to the smooth $(1,1)$-form $\eta_1\wedge\eta_2$ and the point ∞. Recall (cf. [**La**], §II.1 for example) that $g_{\eta_1\wedge\eta_2,\infty}$ is a smooth $\mathbb{C}$-valued function on $Y(\mathbb{C})$ satisfying the following two conditions:

1. Letting ∂ and $\bar\partial : \mathcal{O}_{\mathrm{smooth}}(Y) \longrightarrow \Omega^1_{\mathrm{smooth}}(Y)$ be the usual differential operators defined by

$$\partial f = \frac{\partial f}{\partial z}dz = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)(dx+idy), \quad \bar\partial f = \frac{\partial f}{\partial \bar z}d\bar z = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)(dx-idy),$$

   the function $g_{\eta_1\wedge\eta_2,\infty}$ is a solution of the differential equation

$$\frac{-1}{2\pi i}\partial\bar\partial g_{\eta_1\wedge\eta_2,\infty} = \eta_1\wedge\eta_2.$$

2. The function $g_{\eta_1\wedge\eta_2}$ has at worst logarithmic growth at ∞. More precisely, letting

$$m_{\eta_1\wedge\eta_2} := \int_{X(\mathbb{C})}\eta_1\wedge\eta_2,$$

   the function $g_{\eta_1\wedge\eta_2,\infty}$ can be written as

$$g_{\eta_1\wedge\eta_2,\infty}(x) = -m_{\eta_1\wedge\eta_2}\log|f_\infty(x)|^2 + h(x),$$

   where f_∞ is a rational function on X with a simple zero at ∞ and $h(x)$ is smooth in a neighbourhood of ∞.

The differential

$$\alpha_\infty := \frac{1}{2\pi i}\partial g_{\eta_1\wedge\eta_2,\infty}$$

is a smooth one-form of type $(1,0)$ on $Y(\mathbb{C})$ satisfying

$$d\alpha_\infty = (\partial+\bar\partial)\alpha_\infty = \bar\partial\alpha_\infty = \eta_1\wedge\eta_2.$$

Its Laurent development around ∞ has a principal part of the form

$$\alpha_\infty = \frac{-m_{\eta_1\wedge\eta_2}}{2\pi i}\frac{dz}{z}.$$

This implies that, for any smooth region $R \subset X(\mathbb{C})$ whose boundary does not contain ∞,

$$\int_R \eta_1 \wedge \eta_2 \quad = \quad \int_{\partial R} \alpha_\infty \quad + \quad m_{\eta_1 \wedge \eta_2} \int_R \delta_\infty, \tag{29}$$

where δ_∞ is the Dirac delta current, satisfying

$$\int_R \delta_\infty = \begin{cases} 1 & \text{if } \infty \in R; \\ 0 & \text{if } \infty \notin R. \end{cases}$$

In other words, we have the equality of currents:

$$\eta_1 \wedge \eta_2 = d\alpha_\infty + m_{\eta_1 \wedge \eta_2} \delta_\infty. \tag{30}$$

The following lemma concerning the differential α_∞ shall be used in the proofs of Section 1.2.

Lemma 1.2. — *After setting $X = X_1 = X_2$, viewing $\eta_1 \wedge \eta_2$ as a $(1,1)$-form on X_1, and letting ρ be a closed one-form on X_2, consider the three-form $(\eta_1 \wedge \eta_2) \wedge \rho$ on $X_1 \times X_2$. Let W be any smooth three-chain on $X_1 \times X_2$ which intersects $\{\infty\} \times X_2$ in a smooth one-chain, so that*

$$(\partial W)' := \partial W - \partial(W \cap (\{\infty\} \times X_2))$$

is the complement of a finite set in ∂W. Then the integral

$$\int_{\partial W} \alpha_\infty \wedge \rho := \int_{(\partial W)'} \alpha_\infty \wedge \rho$$

converges, and satisfies

$$\int_W (\eta_1 \wedge \eta_2) \wedge \rho \quad = \quad \int_{\partial W} \alpha_\infty \wedge \rho \quad + \quad m_{\eta_1 \wedge \eta_2} \int_{W \cap (\{\infty\} \times X_2)} \rho.$$

Proof. — Since ρ is closed, equation (30) implies that

$$(\eta_1 \wedge \eta_2) \wedge \rho = d(\alpha_\infty \wedge \rho) + m_{\eta_1 \wedge \eta_2} \cdot \delta_\infty \wedge \rho.$$

Integrating over W and applying Stokes' theorem, we obtain

$$\int_W (\eta_1 \wedge \eta_2) \wedge \rho \quad = \quad \int_{\partial W} \alpha_\infty \wedge \rho \quad + \quad m_{\eta_1 \wedge \eta_2} \int_W \delta_\infty \wedge \rho,$$

and the lemma follows from the fact that $\int_W \delta_\infty \wedge \rho = \int_{W \cap (\{\infty\} \times X_2)} \rho$. □

Following Hain (cf. [**Hain3**, §13.1]), the Hodge filtration on $(I/I^{n+1})^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ is given in terms of (26) by defining $\mathrm{Fil}^j((I/I^{n+1})^\vee \otimes_{\mathbb{Z}} \mathbb{C})$ to be the set of $J \in \mathrm{II}^{\leq n}$ of the form

$$J = \left(\int \omega_1^{(1)} \cdot \omega_2^{(1)} \cdots \omega_{n_1}^{(1)} \right) + \cdots + \left(\int \omega_1^{(t)} \cdot \omega_2^{(t)} \cdots \omega_{n_t}^{(t)} \right)$$

in which each n_i-tuple $(\omega_1^{(i)}, \ldots, \omega_{n_i}^{(i)})$ involves *at least* j smooth differential forms of type $(1,0)$ on $Y(\mathbb{C})$ with logarithmic growth along S. In particular, for $n = 2$ we have

(i) $\mathrm{Fil}^j M_{\mathrm{dR},\mathbb{C}} = M_{\mathrm{dR},\mathbb{C}}$ for all $j \leq 0$;

(ii) The space $\mathrm{Fil}^1 M_{\mathrm{dR},\mathbb{C}}$ is spanned by iterated integrals of the form $J_{\eta_1,\eta_2,\alpha}$ in which α and at least one of η_1 and η_2 lie in $\Omega^1_{\log}(Y(\mathbb{C}))$.

(iii) The space $\mathrm{Fil}^2 M_{\mathrm{dR},\mathbb{C}}$ is spanned by iterated integrals of the form $J_{\omega_1,\omega_2,0}$ in which ω_1 and ω_2 both belong to $\Omega^1_{\log}(Y(\mathbb{C}))$.

(iv) For all $j \geq 3$, $\mathrm{Fil}^j M_{\mathrm{dR},\mathbb{C}} = 0$.

With this description of the Hodge filtration on $M_{\mathrm{dR},\mathbb{C}}$, it is routine to verify that the exact sequence (17) is an exact sequence of filtered vector spaces. The $\mathbb{Z}$-module M_{B} is in fact a prototypical example of an integral mixed Hodge structure. (See [**Hain3**, Cor. 9.3] and [**Hain1**] for a discussion in a more general setting.) Let

$$\kappa_{\mathrm{dR}} \in \mathrm{Ext}^1_{\mathrm{MHS}}(H^1_{\mathrm{B}}(Y)^{\otimes 2}_{\cup=0}, H^1_{\mathrm{B}}(Y)) \tag{31}$$

denote the class of the extension (16) in the category MHS of mixed Hodge structures. We will primarily be concerned with understanding this extension class and finding criteria for it to be non-trivial.

An alternate description of the Hodge filtration

The analytic description of the Hodge filtration on $M_{\mathrm{dR},\mathbb{C}}$ which we have just given is well-suited for the theoretical calculations of this note. However, the difficulties inherent in solving equation (28) seem to make it ill-suited for the practical numerical calculations of Abel-Jacobi maps that are carried out in [**DDLR**]. For the latter, it is convenient to exploit an algebraic description of $M_{\mathrm{dR},\mathbb{C}}$ and its Hodge filtration involving meromorphic differentials of the second kind rather than smooth differentials.

Note that the definitions of $\mathrm{Fil}^j M_{\mathrm{dR},\mathbb{C}}$ given above apply just as well to the algebraic setting, with the exception of $\mathrm{Fil}^1 M_{\mathrm{dR},\mathbb{C}}$ which is described in terms of smooth forms. We now proceed to give a description of this space involving meromorphic differentials on Y of the second kind. Firstly, given $\omega \in \Omega^1_{\log}(Y(\mathbb{C}))$ and a smooth closed one-form η on $Y(\mathbb{C})$, the two-form $\omega \wedge \eta + \eta \wedge \omega$ vanishes identically on Y. In particular, the class of $\omega \otimes \eta + \eta \otimes \omega$ always belongs to $H^1(Y)^{\otimes 2}_{\cup=0}$, even when Y is projective, and the elements of $M_{\mathrm{dR},\mathbb{C}}$ which map to this class are those of the form

$$\begin{aligned} J_{\omega\otimes\eta+\eta\otimes\omega,\alpha}, \qquad & \text{with } d\alpha = 0; \\ J_{\omega\otimes\eta+\eta\otimes\omega,\alpha}(\gamma) := \quad & \textstyle\int_\gamma (\omega\cdot\eta + \eta\cdot\omega) - \int_\gamma \alpha. \end{aligned}$$

A direct calculation reveals that

$$J_{\omega\otimes\eta+\eta\otimes\omega,\alpha}(\gamma) = \left(\int_\gamma \omega\right)\left(\int_\gamma \eta\right) - \int_\gamma \alpha,$$

so that the values of $J_{\omega\otimes\eta+\eta\otimes\omega,\alpha}$ are expressed in terms of products of periods of γ relative to the de Rham cohomology classes of ω, η and α. The function $J_{\omega\otimes\eta+\eta\otimes\omega,\alpha}$ can therefore be readily calculated in practice using meromorphic representatives of the classes of ω, η, and α. It is then clear from the description of the Hodge filtration on $M_{\mathrm{dR},\mathbb{C}}$ given in (ii) above that

$$J_{\omega\otimes\eta+\eta\otimes\omega,\alpha} \text{ belongs to } \mathrm{Fil}^1 M_{\mathrm{dR},\mathbb{C}} \text{ if and only if } \alpha \in \Omega^1_{\log}(Y/\mathbb{C}). \tag{32}$$

Secondly, let us represent a cohomology class $\xi \in H^1_{\mathrm{dR}}(Y/\mathbb{C})$ by a meromorphic differential form η of the second kind on $Y(\mathbb{C})$ with at worst logarithmic poles along S, making use of the algebraic description for $H^1_{\mathrm{dR}}(Y/F)$ given in (23). The principal part of the meromorphic differential 1-form ωF_η at a point $x \in \tilde{Y}$, denoted $\underline{\mathrm{pp}}_x(\omega F_\eta)$, depends only on the image of x

in $Y(\mathbb{C})$. Indeed, let $\gamma x \in \tilde{Y}$, for $\gamma \in \Gamma$, be any other point in $\tilde{Y}$ lying above the same point of $Y(\mathbb{C})$. Comparing principal parts, we find:

$$\underline{\mathrm{pp}}_{\gamma x}(\omega F_\eta) = \underline{\mathrm{pp}}_x(\gamma^*(\omega F_\eta)) = \underline{\mathrm{pp}}_x(\omega F_\eta + c\omega)$$

for some constant c, and the claim follows because ω is regular. In particular, it makes sense to talk about the principal part $\underline{\mathrm{pp}}_P(\omega F_\eta)$ at a point $P \in Y(\mathbb{C})$, even though the one-form ωF_η is only defined on $\tilde{Y}$. A direct application of the Riemann-Roch theorem shows that there exists a meromorphic differential form α on $Y(\mathbb{C})$ with at worst logarithmic poles along S, satisfying

$$\underline{\mathrm{pp}}_P(\alpha) = \underline{\mathrm{pp}}_P(\omega \cdot F_\eta), \qquad \text{for all } P \in Y(\mathbb{C}). \tag{33}$$

The one-form $\omega \cdot F_\eta - \alpha$ is then holomorphic on $\tilde{Y}$. The differentials α satisfying (33) all differ by elements of $\Omega^1_{\log}(Y(\mathbb{C})) = \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathbb{C})$, and we again have that

$$J_{\omega\otimes\eta,\alpha} \text{ belongs to } \mathrm{Fil}^1 M_{\mathrm{dR},\mathbb{C}}. \tag{34}$$

It is then obvious from the definition of $\mathrm{Fil}^1 M_{\mathrm{dR},\mathbb{C}}$ given in (ii) above that this space can be recovered in terms of meromorphic differentials as the vector space generated by the iterated integrals appearing in (32) and (34).
Given a regular differential ω and a meromorphic differential η of the second kind on Y, the task of calculating the meromorphic one-form α satisfying (33) can be carried out explicitly in the case of modular curves, and seems easier in practice than solving the equation (28) associated to a smooth representative of the class of η. See the forthcoming article **[DDLR]** for a more detailed discussion of these computational aspects.

1.2. Points arising from Hodge classes and the fundamental group. — The goal of this section is to recall a construction which goes back at least to **[Ka]** and **[Pu]** whereby the extension class κ_{dR} of (31) arising from the pro-unipotent fundamental group are used to construct rational points in the Jacobian of a (smooth, projective) curve [3]. We retain the same notations as in the previous section and assume that $Y = X \setminus \{\infty\}$ is the complement of a point $\infty \in X(F)$.
A *Hodge class* in a mixed Hodge structure N of weight two is a morphism

$$\xi : \mathbb{Z}(-1) \longrightarrow N \tag{35}$$

of mixed Hodge structures, where $\mathbb{Z}(-1)$ denotes the weight 2 Hodge structure attached to the Tate motive $H^2(\mathbb{P}_1)$. The $\mathbb{Z}$-module of all such Hodge classes is denoted $\mathrm{Hdg}(N)$.
The extension class

$$\kappa_{\mathrm{dR}}(\infty) \in \mathrm{Ext}^1_{\mathrm{MHS}}(H^1_B(X)^{\otimes 2}, H^1_B(X)),$$

of M_{B} in the category of mixed Hodge structures arising from the exact sequence (16) is a natural object of study. We do so by considering, for each class $\xi \in \mathrm{Hdg}(H^1_B(X)^{\otimes 2})$, the element

$$\kappa_{\mathrm{dR}}(\xi;\infty) \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-1), H^1_B(X))$$

3. A tantalising ongoing program (cf. **[Ki]**) also exploits the pro-unipotent fundamental group of a (hyperbolic, affine) curve Y to *bound* its set of integral points, the rough idea being to associate to each point in $Y(\mathbb{Z})$ a system of non-abelian torsors, and then to control the *Selmer varieties* which parametrise them.

associated to the class of the extension $M(\xi) = M(\xi;\infty)$ appearing in the top row of the following commutative diagram with exact rows and cartesian squares:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_B(X) & \longrightarrow & M_B(\xi) & \longrightarrow & \mathbb{Z}(-1) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \xi} & & \\ 0 & \longrightarrow & H^1_B(X) & \longrightarrow & M_B & \longrightarrow & H^1_B(X)^{\otimes 2} & \longrightarrow & 0, \end{array} \tag{36}$$

where the lower exact sequence is (16) applied to Y, once we identify $H^1_B(Y) = H^1_B(X)$ thanks to (19).
There is a canonical isomorphism (cf. e.g. [**Ca**], [**Hain3**, §10.2])

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-1), H^1_{\mathrm{B}}(X)) = \frac{H^1_{\mathrm{dR}}(X/\mathbb{C})}{\Omega^1(X(\mathbb{C})) + H^1_{\mathrm{B}}(X)}. \tag{37}$$

Recalling the exact sequence (17), this identification is obtained by choosing elements

$$v_{\mathrm{Hodge}} \in \mathrm{Fil}^1(M_B(\xi) \otimes \mathbb{C})$$

and $v_{\mathbb{Z}} \in M_B(\xi)$ satisfying

$$\mathrm{p}(v_{\mathrm{Hodge}}) = \mathrm{p}(v_{\mathbb{Z}}) = 1, \tag{38}$$

and assigning to the extension class $M_B(\xi)$ the element

$$v_\xi := i^{-1}(v_{\mathrm{Hodge}} - v_{\mathbb{Z}}) \in H^1_{\mathrm{dR}}(X/\mathbb{C}). \tag{39}$$

Since (38) determines v_{Hodge} and $v_{\mathbb{Z}}$ up to elements of $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(X/\mathbb{C}) = \Omega^1(X/\mathbb{C})$ and $H^1_B(X(\mathbb{C}),\mathbb{Z})$ respectively, the image of v_ξ in the quotient on the right-hand side of (37) depends only on the isomorphism class of the extension $M_B(\xi)$.
The Poincaré pairing

$$H^1_{\mathrm{dR}}(X/F) \times H^1_{\mathrm{dR}}(X/F) \longrightarrow F \tag{40}$$

with $F = \mathbb{C}$ induces an isomorphism

$$\frac{H^1_{\mathrm{dR}}(X/\mathbb{C})}{\Omega^1(X(\mathbb{C})) + H^1_{\mathrm{B}}(X)} \longrightarrow \frac{\Omega^1(X(\mathbb{C}))^\vee}{H^1_{\mathrm{B}}(X)} = \mathrm{Jac}(X)(\mathbb{C}). \tag{41}$$

For all $\xi \in \mathrm{Hdg}(H^1_B(X)^{\otimes 2})$, we continue to use $\kappa_{\mathrm{dR}}(\xi;\infty) \in \mathrm{Jac}(X)(\mathbb{C})$ to describe the canonical point in the jacobian of X arising from the identifications (37) and (41).
The assignment $\xi \mapsto \kappa_{\mathrm{dR}}(\xi;\infty)$ therefore sets up a map

$$\mathrm{Hdg}(H^1_B(X)^{\otimes 2}) \longrightarrow \mathrm{Jac}(X)(\mathbb{C}) \tag{42}$$

sending Hodge classes in $H^1_{\mathrm{dR}}(X)^{\otimes 2}$ to complex points in the Jacobian of X.

Lemma 1.3. — *Given $\infty_1, \infty_2 \in X(\mathbb{C}) - \{o\}$, we have*

$$\kappa_{\mathrm{dR}}(\xi;\infty_1) = \kappa_{\mathrm{dR}}(\xi;\infty_2) + m_\xi((\infty_2) - (\infty_1)),$$

where we recall that $m_\xi = \int_{X(\mathbb{C})} \xi$ is the integral of any smooth two-form representing ξ over the curve $X(\mathbb{C})$ diagonally embedded in $X(\mathbb{C}) \times X(\mathbb{C})$.

Proof. — Let

$$v_\xi(\infty_1) \in H^1_{\mathrm{dR}}(X - \{\infty_1\}/\mathbb{C}), \qquad v_\xi(\infty_2) \in H^1_{\mathrm{dR}}(X - \{\infty_2\}/\mathbb{C})$$

be the elements attached to the extension classes $M_B(\xi)$ with $\infty = \infty_1$ and ∞_2 respectively, following equation (39). The difference $v_\xi(\infty_1) - v_\xi(\infty_2)$ corresponds, modulo elements of $\Omega^1(X)$ and $H^1_B(X(\mathbb{C}), \mathbb{Z})$, to the class in $H^1_{\mathrm{dR}}(X - \{\infty_1, \infty_2\})$ of a regular differential $\omega \in \Omega^1_{\log}(X - \{\infty_1, \infty_2\})$ whose residual divisor is $m_\xi((\infty_2) - (\infty_1))$. But such an element describes the image of the degree zero divisor $m_\xi((\infty_2) - (\infty_1))$ in $\mathrm{Ext}^1(\mathbb{Z}(-1), H^1_{\mathrm{dR}}(X/\mathbb{C}))$, and the result follows. □

It is occasionally convenient to attach an invariant $\kappa_{\mathrm{dR}}(\xi) \in \mathrm{Ext}^1(\mathbb{Z}(-1), H^1_{\mathrm{dR}}(X/\mathbb{C}))$ to an arbitrary element of $\mathrm{Hdg}(H^1_B(X)^{\otimes 2})$ depending only on the pointed curve $(X; o)$ and its fundamental group, and not on the choice of an auxiliary base point $\infty \in X - \{o\}$. We can do this by setting

$$\kappa_{\mathrm{dR}}(\xi) := \kappa_{\mathrm{dR}}(\xi; \infty) + m_\xi((\infty) - (o)). \tag{43}$$

By Lemma 1.3, the right hand side does not depend on the point ∞. It depends only on the Hodge class ξ and the exact sequence (16) arising from the fundamental group $\pi_1(X; o)$. It can be interpreted as arising from the pro-nilpotent fundamental group of $X - \{o\}$ with "tangential base point" at o, in the sense of the article [**De**].
The next two propositions express the points $\kappa_{\mathrm{dR}}(\xi; \infty)$ and $\kappa_{\mathrm{dR}}(\xi)$ in terms of iterated integrals. Let us first introduce a few notations.
Given a Hodge class $\xi \in \mathrm{Hdg}(H^1_B(X)^{\otimes 2})$, choose a representative

$$\tilde{\xi} = \sum_{j=1}^{t} \omega_j \otimes \eta_j,$$

for its de Rham cohomology class, in such a way that ω_j and η_j are smooth closed one-forms on X for each j, at least one of which is of type $(1, 0)$ (and hence, holomorphic on X). If $\rho \in \Omega^1(X)$ is any holomorphic differential, let $\gamma_\rho \in \mathbb{C}[\Gamma_{\mathrm{ab}}]$ be a path whose homology class is Poincaré dual to ρ, i.e., such that

$$\int_{\gamma_\rho} \tilde{\eta} = \langle \rho, \eta \rangle, \quad \text{for all } \eta \in H^1_{\mathrm{dR}}(X/\mathbb{C}).$$

Proposition 1.4. — *With notations as above, let α_∞ be a smooth differential on $Y(\mathbb{C})$ of type $(1, 0)$, with at worst logarithmic poles at ∞, satisfying*

$$d\alpha_\infty = \sum_{j=1}^{t} \omega_j \wedge \eta_j \ \text{ on } Y(\mathbb{C}).$$

Then, for all $\rho \in \Omega^1(X/\mathbb{C})$,

$$\kappa_{\mathrm{dR}}(\xi;\infty)(\rho) = \int_{\gamma_\rho}\left(\sum_{j=1}^{t}\omega_j\cdot\eta_j - \alpha_\infty\right) \quad \textit{and}$$

$$\kappa_{\mathrm{dR}}(\xi)(\rho) = \int_{\gamma_\rho}\left(\sum_{j=1}^{t}\omega_j\cdot\eta_j - \alpha_\infty\right) + m_\xi\int_o^\infty \rho.$$

Proof. — The first equality is a direct consequence of the definitions, using the fact that the class of the extension $M_B(\xi;\infty)$ attached to the pair (ξ,∞) is described by the iterated integral $J_{\tilde{\xi},\alpha_\infty}$. The second equality then follows from equation (43) defining $\kappa_{\mathrm{dR}}(\xi)$. □

We conclude this section with a formula analogous to the one in Proposition 1.4, expressing $\kappa_{\mathrm{dR}}(\xi;\infty)$ and $\kappa_{\mathrm{dR}}(\xi)$ in terms of differentials of the second kind. This formula turns out to be particularly useful in computing the points $\kappa_{\mathrm{dR}}(\xi)$ numerically, as it is done in **[DDLR]** for Hodge classes arising from Hecke correspondences on a classical elliptic modular curve. Choose a representative

$$\tilde{\xi} = \sum_{j=1}^{s}\omega_j^+\otimes\eta_j^+ + \sum_{j=1}^{t}\eta_j^-\otimes\omega_j^- \in \Omega^1(X/\mathbb{C})\otimes\Omega^1_{\mathrm{mer}}(X/\mathbb{C}) + \Omega^1_{\mathrm{mer}}(X/\mathbb{C})\otimes\Omega^1(X/\mathbb{C})$$

of the class ξ, in such a way that the elements $\omega_j^\pm$ belong to $\Omega^1(X) = \mathrm{Fil}^1 H^1_{\mathrm{dR}}(X/\mathbb{C})$ and the differentials $\eta_j^\pm$ are of the second kind and regular at ∞.

Proposition 1.5. — *With notations as above, let* α_∞ *be a meromorphic differential on* X*, with at worst logarithmic poles at* ∞*, satisfying*

$$\underline{\mathrm{pp}}_x(\alpha_\infty) = \sum_{j=1}^{s}\underline{\mathrm{pp}}_x(\omega_j^+ F_{\eta_j^+}) - \sum_{j=1}^{t}\underline{\mathrm{pp}}_x(\omega_j^- F_{\eta_j^-}), \qquad \textit{for all } x\in Y(\mathbb{C}).$$

Then, for all $\rho\in\Omega^1(X/\mathbb{C})$,

$$\kappa_{\mathrm{dR}}(\xi;\infty)(\rho) = \int_{\gamma_\rho}\left(\sum_{j=1}^{s}\omega_j^+\cdot\eta_j^+ + \sum_{j=1}^{t}\eta_j^-\cdot\omega_j^- - \alpha_\infty\right).$$

Proof. — The proof is the same as for Proposition 1.4, using this time the algebraic description of the Hodge filtration on $M_{\mathrm{dR},\mathbb{C}}$ in terms of differentials of the second kind. □

2. Points arising from algebraic cycles

Hodge classes in $H^1_{\mathrm{B}}(X)^{\otimes 2}$ can be constructed naturally from geometry, by exploiting the *cycle class maps*

$$\mathrm{cl} : \mathrm{Pic}(X\times X) \longrightarrow \mathrm{Hdg}(H^2_B(X\times X)).$$

More precisely, let ε be the projector on $\mathrm{Pic}(X\times X)$ given by the rule

$$\varepsilon(Z) = Z - i_{1,*}\pi_{1,*}(Z) - i_{2,*}\pi_{2,*}(Z),$$

where

$$\pi_1, \pi_2 : X \times X \longrightarrow X, \qquad \pi_1(P_1, P_2) = P_1, \quad \pi_2(P_1, P_2) = P_2$$

are the natural projections onto the first and second factors, and

$$i_1, i_2 : X \hookrightarrow X \times X, \qquad i_1(P) = (P, o), \quad i_2(P) = (o, P)$$

are the natural inclusions of X into the horizontal and vertical copies of X in $X \times X$ over the point o.

The idempotent ε annihilates the factors $H^2(X) \otimes H^0(X)$ and $H^0(X) \otimes H^2(X)$ in the Künneth decomposition of $H^2(X^2)$ (for any of the cohomology theories, Betti, de Rham or étale). For any $Z \in \mathrm{Pic}(X \times X)(F)$ we can thus define

$$\xi_Z := \mathrm{cl}(\varepsilon Z) \in \mathrm{Hdg}(H^1_B(X, \mathbb{Z})^{\otimes 2}). \tag{44}$$

In this way, we can assign to the cycle $Z \subset X \times X$ a point

$$\kappa_{\mathrm{dR}}(\xi_Z) \in \mathrm{Jac}(X)(\mathbb{C}),$$

The goal of this section is to associate to any $Z \in \mathrm{Pic}(X \times X)(F)$ a point

$$P_Z \in \mathrm{Jac}(X)(F),$$

in such a way that for all complex embeddings $\iota : F \longrightarrow \mathbb{C}$,

$$\kappa_{\mathrm{dR}}(\xi_Z) = P_Z \times_\iota \mathrm{Spec}(\mathbb{C}).$$

Let us describe a geometric recipe for constructing the desired point P_Z. Given a smooth projective variety $V_{/F}$ and a positive integer $c \geq 1$, let

$$\mathrm{CH}^c(V) = \mathcal{Z}^c(V) / \sim_{\mathrm{rat}}$$

denote the abelian group of rational equivalence classes of cycles of codimension c in V. We regard it as a functor from the category of field extensions of F in $\mathbb{C}$ to the category of abelian groups, given by the rule

$$K/F \ \mapsto \ \mathrm{CH}^c(V)(K) := \{[Z] : \ Z \in \mathcal{Z}^c(V), \quad \text{such that } {}^\sigma Z \sim_{\mathrm{rat}} Z \ \text{ for all } \sigma \in \mathrm{Aut}_K(\mathbb{C})\}.$$

For any integer $i \geq 1$, let $X_i = X$ be a copy of the curve X and for any subset $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ with $m \leq n$ of indices in a given finite set, let $X_{i_1, \ldots, i_m}$ denote a copy of the curve X embedded in $X_1 \times \cdots \times X_n$ via the map

$$x \mapsto (x_i)_{i=1}^n, \qquad \text{where } x_i = \begin{cases} x & \text{if } i \in \{i_1, ..., i_m\}, \\ o & \text{otherwise.} \end{cases}$$

The one-dimensional cycle $Z \subset X_1 \times X_2$ gives rise to the following divisor classes on X:

$$D_{12} = Z \cdot X_{12}, \quad D_1 = Z \cdot X_1, \quad D_2 = Z \cdot X_2.$$

Set

$$P_Z = D_{12} - \deg(D_{12}) \cdot o \quad \in \quad \mathrm{Pic}^0(X) = \mathrm{Jac}(X). \tag{45}$$

We now give an alternate geometric description of P_Z which will be used to relate this point to iterated integrals. Let

$$\Pi_Z := Z_{12} \times X_{34} \subset X_1 \times X_2 \times X_3 \times X_4,$$

where Z_{12} denotes a copy of the cycle Z embedded in $X_1 \times X_2$. The class of the cycle Π_Z belongs to $\mathrm{CH}^2(X_1 \times X_2 \times X_3 \times X_4)$, and induces a morphism

$$\Pi_Z : \mathrm{CH}^2(X_1 \times X_2 \times X_3) \longrightarrow \mathrm{CH}^1(X_4)$$

by the usual rule

$$\Pi_Z(\Delta) = \pi_{4*}(\Pi_Z \cdot \pi_{123}^*\Delta), \tag{46}$$

where

$$\pi_4 : X_1 \times X_2 \times X_3 \times X_4 \longrightarrow X_4, \qquad \pi_{123} : X_1 \times X_2 \times X_3 \times X_4 \longrightarrow X_1 \times X_2 \times X_3$$

denote the natural projections onto the fourth and first three factors respectively.
The Gross-Kudla-Schoen *modified diagonal cycle* in $X^3 = X_1 \times X_2 \times X_3$ is defined to be

$$\Delta_{\mathrm{GKS}} = X_{123} - X_{23} - X_{13} - X_{12} + X_1 + X_2 + X_3 \in \mathrm{CH}^2(X_1 \times X_2 \times X_3). \tag{47}$$

Lemma 2.1. — *For all* $Z \in \mathrm{Pic}(X \times X)$,

$$P_Z = \Pi_Z(\Delta_{\mathrm{GKS}}).$$

Proof. — A direct calculation using only the definition (46) shows that

$$\Pi_Z(X_{123}) = D_{12},$$
$$\Pi_Z(X_{23}) = D_2, \quad \Pi_Z(X_{13}) = D_1, \quad \Pi_Z(X_{12}) = \deg(D_{12}) \cdot o,$$
$$\Pi_Z(X_1) = \deg(D_1) \cdot o, \quad \Pi_Z(X_2) = \deg(D_2) \cdot o, \quad \Pi_Z(X_3) = 0.$$

The lemma follows. □

Remark 2.2. — Let ∞ be any point in X (possibly equal to o). Viewing $X_1 \times \{\infty\}$ and $\{\infty\} \times X_2$ as divisors in $X_1 \times X_2$, a direct calculation shows that

$$P_{X_1 \times \{\infty\}} = P_{\{\infty\} \times X_2} = 0.$$

Hence $P_Z = P_{\varepsilon Z}$ and the point P_Z depends only on the Hodge class $\xi_Z \in \mathrm{Hdg}(H_B^1(X)^{\otimes 2})$.

We will use Lemma 2.1 to derive an analytic formula for the point P_Z, or rather, for its image under the Abel-Jacobi map

$$\mathrm{AJ}_X : \mathrm{CH}^1(X)_0 \longrightarrow \frac{\Omega^1(X/\mathbb{C})^\vee}{H_1(X(\mathbb{C}), \mathbb{Z})}.$$

This Abel-Jacobi map is defined on degree 0 divisors D by the familiar rule:

$$\mathrm{AJ}_X(D)(\omega) = \int_{\partial^{-1}(D)} \omega, \tag{48}$$

where $\partial^{-1}(D)$ is any smooth one-chain on $X(\mathbb{C})$ whose boundary is D. It admits a higher dimensional generalisation

$$\mathrm{AJ}_{X^3} : \mathrm{CH}^2(X^3)_0(\mathbb{C}) \longrightarrow J^2(X^3), \tag{49}$$

where $\mathrm{CH}^2(X^3)_0$ is the Chow group of *null-homologous* cycles on X^3, and

$$J^2(X^3) = \frac{\mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3/\mathbb{C})^\vee}{H_3(X^3(\mathbb{C}), \mathbb{Z})}$$

is the intermediate Jacobian associated to $H^3(X^3)$. This Abel-Jacobi map is defined by the rule generalising (48),

$$\mathrm{AJ}_{X^3}(\Delta)(\omega) = \int_{\partial^{-1}(\Delta)} \omega, \qquad \omega \in \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3/\mathbb{C}). \tag{50}$$

The following lemma verifies that the Gross-Kudla-Schoen cycle Δ_{GKS} lies in the domain of this Abel-Jacobi map.

Lemma 2.3. — *The cycle* Δ_{GKS} *is null-homologous.*

Proof. — The obvious method to prove this lemma is to exploit the Künneth decomposition of the cohomology of $X_1 \times X_2 \times X_3$, as it is done in **[GrSc]** for example. We give a more detailed, direct argument, which yields a differentiable 3-chain on $X_1 \times X_2 \times X_3$ having Δ_{GKS} as boundary. While not strictly necessary to prove Lemma 2.3, such an explicit 3-chain will be used in subsequent calculations.
Let g be the genus of the projective curve X, and let $\gamma_1, \ldots, \gamma_{2g}$ be a collection of paths in $\mathbb{P}(X; o)$ whose associated homology classes, denoted $[\gamma_1], \ldots, [\gamma_{2g}]$, form an integral basis for $H_1(X(\mathbb{C}), \mathbb{Z})$. Let $\gamma_1^*, \ldots, \gamma_{2g}^*$ be another collection of paths, whose homology classes give rise to the dual basis in $H_1(X(\mathbb{C}), \mathbb{Z})$ relative to the intersection product. In other words, we require that

$$([\gamma_i] \cdot [\gamma_j^*]) = \delta_{ij}, \qquad \text{for all } 1 \leq i, j \leq 2g,$$

where δ_{ij} is the Kronecker delta. Let Δ denote the diagonal in $X \times X$. Its class in $H_2(X \times X)$ is given by

$$\mathrm{cl}(\Delta) = \mathrm{cl}(X \times \{o\}) + \mathrm{cl}(\{o\} \times X) + \sum_{i=1}^{2g} [\gamma_i] \otimes [\gamma_i^*],$$

and therefore there is a smooth 3-chain $W \subset X \times X$ such that

$$\Delta = (X \times \{o\}) + (\{o\} \times X) + \sum_{i=1}^{2g} \gamma_i \times \gamma_i^* + \partial(W). \tag{51}$$

Applying this relation to X_{123}, viewed as the diagonal in $X_{12} \times X_3$, gives

$$X_{123} = X_{12} + X_3 + \sum_{i=1}^{2g} \gamma_i(12) \times \gamma_i^*(3) + \partial(W_{12,3}), \tag{52}$$

where

1. For $\{r, s\} \in \{1, 2, 3\}$, the regions $\gamma_i(rs)$ or $\gamma_i(r)$ are the paths γ_i diagonally embedded in $X_{rs} \subset X_1 \times X_2 \times X_3$ and in X_r respectively, and similar conventions are adopted for the paths γ_j^*.
2. The three-chain $W_{12,3}$ is a copy of the region W of (51), embedded in $X_{12} \times X_3$ in the natural way.

By an analogous reasoning, we find

$$X_{13} = X_1 + X_3 + \sum_{i=1}^{2g} \gamma_i(1) \times \gamma_i^*(3) + \partial(W_{13}), \tag{53}$$

and

$$X_{23} = X_2 + X_3 + \sum_{i=1}^{2g} \gamma_i(2) \times \gamma_i^*(3) + \partial(W_{23}), \tag{54}$$

with the obvious meaning given to $W_{13} \subset X_1 \times \{o\} \times X_3$ and to $W_{23} \subset \{o\} \times X_2 \times X_3$. Subtracting (53) and (54) from (52), we obtain

$$\Delta_{\mathrm{GKS}} = \sum_{i=1}^{2g} (\gamma_i(12) - \gamma_i(1) - \gamma_i(2)) \times \gamma_i^*(3) + \partial(W_{12,3} - W_{13} - W_{23}).$$

But

$$\gamma_i(12) - \gamma_i(1) - \gamma_i(2) = \partial(T_{\gamma_i}(12)),$$

where

$$T_\gamma(12) = \{(\gamma(r), \gamma(s)), \quad 0 \leq s \leq r \leq 1\} \subset X_1 \times X_2$$

is the two-dimensional region introduced in (25). Therefore,

$$\Delta_{\mathrm{GKS}} = \partial \left(\sum_{i=1}^{2g} T_{\gamma_i}(12) \times \gamma_i^*(3) + W_{12,3} - W_{13} - W_{23} \right), \tag{55}$$

and the result follows. □

Lemma 2.4. — *For all $Z \in \mathrm{CH}^1(X \times X)$, and all $\rho \in \Omega^1(X)$,*

$$\mathrm{AJ}_X(P_Z)(\rho) = \mathrm{AJ}_{X^3}(\Delta_{\mathrm{GKS}})(\xi_Z(12) \wedge \rho(3)),$$

where the classes

$$\xi_Z(12) \in \mathrm{Fil}^1 H^2_{\mathrm{dR}}(X_1 \times X_2/\mathbb{C}), \qquad \rho(3) \in \Omega^1(X_3) = \mathrm{Fil}^1 H^1_{\mathrm{dR}}(X_3/\mathbb{C})$$

are obtained in the natural way from the classes ξ and ρ respectively (so that, in particular, $\xi_Z(12) \wedge \rho(3)$ belongs to $\mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3/\mathbb{C})$).

Proof. — By Lemma 2.1 and the functoriality properties of the Abel-Jacobi map under correspondences,

$$\mathrm{AJ}_X(P_Z)(\rho) = \mathrm{AJ}_X(\Pi_Z \Delta_{\mathrm{GKS}})(\rho) = \mathrm{AJ}_{X^3}(\Delta_{\mathrm{GKS}})(\Pi_Z^* \rho),$$

where

$$\Pi_Z^* : H^1_{\mathrm{dR}}(X_4/\mathbb{C}) \longrightarrow H^3_{\mathrm{dR}}(X_1 \times X_2 \times X_3/\mathbb{C})$$

is the morphism induced on de Rham cohomology by the correspondence $\Pi_Z = Z_{12} \times X_{34}$. A direct calculation reveals that

$$\Pi_Z^*(\rho) = \mathrm{cl}(Z) \wedge \rho,$$

and therefore

$$\mathrm{AJ}_X(P_Z)(\rho) = \mathrm{AJ}_{X^3}(\Delta_{\mathrm{GKS}})(\mathrm{cl}(Z) \wedge \rho).$$

By a similar reasoning,

$$\mathrm{AJ}_X(P_{\varepsilon Z})(\rho) = \mathrm{AJ}_{X^3}(\Delta_{\mathrm{GKS}})(\xi_Z \wedge \rho),$$

and the result follows from Remark 2.2. □

For the next proposition, we adopt the same notations that were introduced in the statement of Proposition 1.4 and the discussion immediately preceding it.

Theorem 2.5. — *For all* $\xi \in \mathrm{Hdg}(H^1_B(X)^{\otimes 2})$ *and all* $\rho \in \Omega^1(X/\mathbb{C})$,

$$\mathrm{AJ}_{X^3}(\Delta_{\mathrm{GKS}})(\xi(12) \wedge \rho(3)) = \int_{\gamma_\rho} \left(\sum_{j=1}^{t} \omega_j \cdot \eta_j - \alpha_\infty \right) + m_\xi \int_o^\infty \rho.$$

Proof. — By (55), and the definition of the Abel-Jacobi map,

$$\mathrm{AJ}(\Delta_{\mathrm{GKS}})(\xi \wedge \rho) = \sum_{i=1}^{2g} \int_{T_{\gamma_i}(12) \times \gamma_i^*(3)} \tilde{\xi} \wedge \rho \quad + \int_{W_{12,3}} \tilde{\xi} \wedge \rho, \tag{56}$$

where we have used the fact that $\tilde{\xi} \wedge \rho$ vanishes identically on $W_{13} \subset X_1 \times \{o\} \times X_3$ and on $W_{23} \subset \{o\} \times X_2 \times X_3$ to dispense with the integrals over these two regions. The first term appearing in the right hand side of (56) is equal to

$$\sum_{i=1}^{2g} \int_{T_{\gamma_i}(12) \times \gamma_i^*(3)} \tilde{\xi} \wedge \rho = \sum_{i=1}^{2g} \int_{T_{\gamma_i}(12)} \tilde{\xi} \int_{\gamma_i^*(3)} \rho.$$

This last expression can also be viewed as the integral of the two-form $\tilde{\xi}$ over the 2-chain

$$\sum_{i=1}^{2g} \left(\int_{\gamma_i^*} \rho \right) T_{\gamma_i} = T_{\gamma_\rho},$$

where

$$T_{\gamma_\rho} = \sum_{i=1}^{2g} \lambda_i T_{\gamma_i}, \qquad \text{with } [\gamma_\rho] = \sum \lambda_i [\gamma_i].$$

By the formula for the iterated integral that is given in (25), it follows that

$$\sum_{i=1}^{2g} \int_{T_{\gamma_i}(12) \times \gamma_i^*(3)} \tilde{\xi} \wedge \rho = \int_{\gamma_\rho} \sum_{j=1}^{t} \omega_j \cdot \eta_j. \tag{57}$$

To deal with the second term in the right-hand side of (56), we choose an auxiliary point ∞ which is not in the image of any of the paths γ_i and γ_i^*, and such that $W_{12,3} \cap ((\infty, \infty) \times X_3)$ is a smooth one-chain. By Lemma 1.2,

$$\int_{W_{12,3}} \tilde{\xi} \wedge \rho = \int_{\partial W_{12,3}} \alpha_\infty \wedge \rho + m_\xi \int_{W_{12,3} \cap ((\infty,\infty) \times X_3)} \rho. \tag{58}$$

Recall that

$$\partial W_{12,3} = X_{123} - X_{12} - X_3 - \sum_{i=1}^{2g} \gamma_i(12) \times \gamma_i^*(3),$$

and note that the differential $\alpha_\infty \wedge \rho$ vanishes identically on X_{123}, X_{12} and X_3 since it is of type $(2, 0)$. Therefore, by a similar argument as above,

$$\int_{\partial W_{12,3}} \alpha_\infty \wedge \rho = -\sum_{i=1}^{2g} \int_{\gamma_i} \alpha_\infty \int_{\gamma_i^*} \rho = -\int_{\gamma_\rho} \alpha_\infty. \tag{59}$$

Furthermore, the region of integration that appears in the second term of (58) has boundary

$$\begin{aligned}\partial(W_{12,3} \cap ((\infty,\infty) \times X_3)) &= \left(X_{123} - X_{12} - X_3 - \sum_{i=1}^{2g} \gamma_i \times \gamma_i^* \right) \cap ((\infty,\infty) \times X_3) \\ &= (\infty,\infty,\infty) - (\infty,\infty,o).\end{aligned}$$

It follows that

$$m_\xi \int_{W_{12,3} \cap ((\infty,\infty)\times X_3)} \rho = m_\xi \int_o^\infty \rho. \tag{60}$$

Substituting (59) and (60) into (58) yields

$$\int_{W_{12,3}} \tilde{\xi} \wedge \rho = -\int_{\gamma_\rho} \alpha_\infty + m_\xi \int_o^\infty \rho. \tag{61}$$

Finally, using (57) and (61) to rewrite the right hand side of (56), we obtain

$$\mathrm{AJ}(\Delta_{\mathrm{GKS}})(\xi \wedge \rho) = \int_{\gamma_\rho} \sum_{j=1}^{t} \omega_j \cdot \eta_j - \int_{\gamma_\rho} \alpha_\infty + m_\xi \int_o^\infty \rho, \tag{62}$$

as was to be shown. □

Corollary 2.6. — *For all $Z \in \mathrm{Pic}(X \times X)$, and all $\rho \in \Omega^1(X)$,*

$$\mathrm{AJ}_X(P_Z)(\rho) = \int_{\gamma_\rho} \left(\sum_{j=1}^{t} \omega_j \cdot \eta_j - \alpha_\infty \right) + m_{\xi_Z} \int_o^\infty \rho,$$

where $\tilde{\xi}_Z = \sum_{i=1}^t \omega_i \otimes \eta_i$ is a representative of $\xi_Z := \mathrm{cl}(\varepsilon Z)$ and α_∞ and γ_ρ are as in the statement of Theorem 2.5.

Proof. — This follows directly from Lemma 2.4 and Theorem 2.5. □

By combining Proposition 1.4 and Corollary 2.6 we obtain the sought-for relation between the extension classes $\kappa_{\mathrm{dR}}(\xi)$ arising from $\pi_1(X;o)$ and the points P_Z.

Theorem 2.7. — *For all $Z \in \mathrm{Pic}(X \times X)$, $\kappa_{\mathrm{dR}}(\xi_Z) = P_Z$ in $\mathrm{Jac}(X)(\mathbb{C})$.*

In light of Lemma 2.1, this is Theorem 1. As anticipated in the introduction, we can recover from it a result that is equivalent to the one obtained by Kaenders in [**Ka**]:

Corollary 2.8. — *Let $X_{12} \in \mathrm{Pic}(X \times X)$ be the class of the diagonal divisor in $X \times X$ up to rational equivalence. Then $\kappa_{\mathrm{dR}}(\xi_{X_{12}}) = (2g-2)o - K$ in $\mathrm{Jac}(X)(\mathbb{C})$, where $K \in \mathrm{Pic}(X)$ is the canonical divisor on X.*

Proof. — By Theorem 2.7 and (45), $\kappa_{\mathrm{dR}}(\xi_{X_{12}}) = P_{X_{12}} = X_{12} \cdot X_{12} - \deg(X_{12} \cdot X_{12})o$. An easy application of the Riemann-Roch theorem and the adjunction formula shows that the self-intersection of the diagonal in the square of a curve is $-K$. The corollary follows. □

3. Modular and Shimura curves

In this last section we specialize to the setting where X is a Shimura curve over a totally real field.

Let F be a totally real number field of degree d, and R_F its ring of integers. Fix a classical Hilbert modular newform g of parallel weight 2, level N_g and trivial nebentypus. We refer the reader to [**Zh**] for the precise definitions of Shimura curves, Hecke correspondences and L-functions that we use throughout this section.

Choose a square-free ideal $D = \wp_1 \cdot ... \cdot \wp_r \,||\, N_g$ of R_F dividing it exactly (by which we mean that D and N_g/D are relatively prime). Assume that $r + d$ is *odd*. (If d is odd, a possible choice for D is thus $D = R_F$).

Choose an ideal N of R_F such that $N_g \mid N$ and $D \,||\, N$, and put $M = N/D$. Let $X_0^D(M)$ denote the Shimura curve associated with the Eichler order of level M in a quaternion algebra ramified precisely at D and at all archimedean places of F but one, and write $J_0^D(M)$ for its jacobian.

Let $\mathbb{T} := \langle T_\wp, \wp \nmid N \rangle_{\mathbb{Q}}$ be the Hecke algebra generated by the set of Hecke operators at primes $\wp$ not dividing N, acting on $J_0^D(M)$. This $\mathbb{Q}$-algebra is isomorphic to a free product of number fields, namely

$$\mathbb{T} \simeq \prod_{D||n|N} \prod_h K_h \tag{63}$$

where for each ideal n such that $D \,||\, n$ and $n \mid N$, h runs through a set of representatives for the orbits of $\mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q})$ acting on the set of Hilbert newforms of level n, and K_h is the number field generated by the eigenvalues of $\mathbb{T}$ acting on h. In particular $\mathbb{T}$ contains a copy of K_g; write $T_g \in \mathbb{T}$ for the corresponding idempotent in the decomposition (63).

If M is prime then in fact $\mathbb{T} \simeq \mathrm{End} J_0^D(M)$. But as soon as M is a composite ideal, the endomorphism algebra of $J_0^D(M)$ is larger, namely a product of matrix algebras:

$$\mathrm{End}^0(J_0^D(M)) := \mathrm{End}(J_0^D(M)) \otimes \mathbb{Q} \simeq \prod_{D||n|N} \prod_h M_{\sigma(N/n)}(K_h), \tag{64}$$

where $\sigma(N/n)$ is the number of ideal dividing N/n. Its center is thus $\mathbb{T}$.

Note that $\mathrm{End}^0(J_0^D(M))[g] := T_g \cdot \mathrm{End}^0(J_0^D(M)) \simeq M_{\sigma(N/N_g)}(K_g)$, the center of which is $\mathbb{T}[g] := T_g \cdot \mathbb{T}$.

Since

$$\mathrm{End}(J_0^D(M)) \simeq \frac{\mathrm{Pic}(X_0^D(M) \times X_0^D(M))}{\pi_1^*(\mathrm{Pic}(X_0^D(M)) + \pi_2^*(\mathrm{Pic}(X_0^D(M))} \tag{65}$$

by [**LaBi**, Theorem 11.5.1], we can define

$$H_g := \{\xi_T, T \in \mathbb{T}[g]\} \subseteq \mathrm{Hdg}(H^1(X_0^D(M))^{\otimes 2} \otimes \mathbb{Q}) \tag{66}$$

to be the module of Hodge classes associated with g.

Definition 3.1. — $\underline{P}_g := \kappa_{\mathrm{dR}}(H_g) \subseteq \mathbb{Q} \otimes J_D^0(M)(F)$.

The Hodge classes in H_g can be written down explicitly. For instance, a representative for $\xi_g := \xi_{T_g} \in H_g$ is given as follows: let $\{\omega_{g,i}\}$ be a basis of the g-isotypical component of

the module of regular differential forms on $X_0^D(M)$ and complete it to a symplectic basis $\{\omega_{g,i}, \eta_{g,i}\}$ of $H^1_{\mathrm{dR}}(X_0^D(M))[g]$. Then

$$\tilde{\xi}_g = \sum_i \omega_{g,i} \otimes \eta_{g,i} - \eta_{g,i} \otimes \omega_{g,i} \tag{67}$$

is a differential form on $X_0^D(M) \times X_0^D(M)$, rational over F, which represents ξ_g. The point $P_g := \kappa_{\mathrm{dR}}(\xi_g) \in \underline{P}_g$ can then be computed as the tuple of periods of the iterated integral associated with (67); similar formulas can be obtained of course for a complete basis of $\underline{P}_g$: see [**DDLR**] for explicit calculations.

Recall that $\kappa_{\mathrm{dR}}(\xi)$ depends on the choice of the base point o. If we are dealing with the classical elliptic modular curve associated to the split algebra $\mathrm{M}_2(\mathbb{Q})$, we can take o to be the cusp at infinity, which is a rational point on $X_0(M)$.

If $X_0^D(M)$ is the Shimura curve attached to an order in a *division* quaternion algebra over an arbitrary totally real number field F, then $X_0^D(M)(F)$ may be empty: see e.g. [**Sh75**, Theorem 5.1], which establishes criteria for the existence of real points on these curves and shows in particular that $X_0^D(M)(\mathbb{R}) = \emptyset$ when $F = \mathbb{Q}$. Instead of fixing a single point o rational over F, one proceeds as in [**GrKu**, §13], [**YZZ**] and [**Zh**] by rather considering a canonical $\mathrm{Gal}\,(\bar{F}/F)$-invariant divisor class $o = \sum a_i o_i$ of degree 1 with rational coefficients on $X_0^D(M)$, which in [**Zh**] is called the *Hodge class* of $X_0^D(M)$. In this case, by $\kappa_{\mathrm{dR}}(\xi)$ we mean $\sum a_i \kappa_{\mathrm{dR}}(\xi; o_i)$.

Choose now another normalised Hilbert newform f of weight 2, level $N_f \mid N$ and trivial nebentypus, different from g. The triple product L-function $L(g \otimes g \otimes f, s)$, which was first studied by Garrett in [**Gar**], is the L-function associated with the tensor product $V_g \otimes V_g \otimes V_f$, where V_g and V_f denote the two-dimensional ℓ-adic representations of $G_F := \mathrm{Gal}\,(\bar{F}/F)$ attached to g and f respectively. It is explicitly given by an Euler product with Euler factors of degree 8 at the primes $\wp \nmid N$, and factors as

$$L(g \otimes g \otimes f, s) = L(f, s-1) L(\mathrm{Sym}^2 g \otimes f, s), \tag{68}$$

mirroring the decomposition

$$V_g \otimes V_g = \mathbb{Q}_\ell(-1) \oplus \mathrm{Sym}^2 V_g$$

of ℓ-adic representations of G_F.

Thanks to the work of Garrett, Piatetski-Shapiro–Rallis, Harris–Kudla, Prasad and others (cf. [**Gar**],[**Pr**] and the references therein), it is known that $L(g \otimes g \otimes f, s)$ admits analytic continuation to $\mathbb{C}$ and satisfies a functional equation relating its values at s and $4 - s$. The sign ϵ in this functional equation is expressed as a product

$$\epsilon = (-1)^d \prod_{\wp \mid \gcd(N_g, N_f)} \epsilon_\wp \tag{69}$$

of local signs $\epsilon_\wp \in \{\pm 1\}$; for primes $\wp$ dividing exactly both N_g and N_f,

$$\epsilon_\wp = -a_\wp(f)$$

is the negative of the eigenvalue of the Hecke operator $T_\wp$ acting on f (cf. [**GrKu**], [**Pr**]), and in particular does not depend on the choice of g. When $\wp^2 \mid N_g$ or N_f, the value of $\epsilon_\wp$ depends

on the local behavior of the automorphic representations associated with g and f; see [**Pr**] for more details. Assume that

$$\epsilon(f) = -1, \qquad \epsilon(\mathrm{Sym}^2(g) \otimes f) = +1. \tag{70}$$

Then $\epsilon = -1$ and we can set $D = D_{g;f} = \prod_{\epsilon_\wp = -1} \wp$; note that D is the product of $r = \sharp\{\epsilon_\wp = -1\}$ distinct prime ideals, and that $r + d$ is odd by (69). Assume that $D \,||\, N$ and put as above $M = M_{g;f} = N/D$.
The quotient of $J_0^D(M)$ associated to f by the theory of Eichler-Shimura is an abelian variety which is well-defined only up to F-isogenies, and is isogenous to the power $A_f^{\sigma(N/N_f)}$ of an F-simple abelian variety A_f. This means that there are $\sigma(N/N_f)$ F-linear independent modular parametrizations

$$\pi_f^d : J_0^D(M) \longrightarrow A_f$$

of A_f, one for each divisor d of N/N_f.

Definition 3.2. — Let

$$\underline{P}_{g,f} := \sum_{d|\frac{N}{N_f}} \pi_f^d(\underline{P}_g) \subseteq A_f(F) \otimes \mathbb{Q} \tag{71}$$

be the subspace of the Mordell-Weil group (tensored with $\mathbb{Q}$) of A_f associated to the pair of modular forms (g, f).

In particular, if we let $\pi_f : J_D^0(M) \to J_0^D(N_f/D) \to A_f$ denote the map which factors through the natural projection of $J_0^D(M)$ onto $J_0^D(N/N_f)$, the point

$$P_{g,f} = \pi_f(P_g) \tag{72}$$

belongs to this space, and can be explicitly computed by means of (67).
Formula (72), combined with Theorem 2.5, provides a suggestive construction of a point on a modular abelian variety by means of iterated integrals. Theorem 1 can be used to show that, in the particular case that g has *rational* fourier coefficients, $P_{g,f}$ is essentially equal to a point that was first introduced by S. Zhang by a different method, whose definition we recall here for the convenience of the reader:

Definition 3.3. — [S. Zhang] Assume g has fourier coefficients in $\mathbb{Z}$. Let E_g/F be the elliptic curve corresponding to g by the Eichler-Shimura construction and $\pi_g : X_0^D(M) \longrightarrow E_g$ be a modular parametrization *of curves*. Letting $O \in E_g(F)$ denote the identity element, define

$$P_{g,f}^{Zh} := \pi_{f,*}\pi_g^*(O).$$

We refer the reader to the forthcoming Ph. D thesis [**Daub**] of M. Daub for more details concerning the connection between (72) and Definition 3.3. See also W. Stein's appendix to [**DDLR**] for the effective numerical computation of the points $P_{g,f}^{Zh}$ in the classical setting of elliptic modular curves over $F = \mathbb{Q}$. Note that Definition 3.3 does not readily generalize to the case in which the fourier coefficients of g are not integral, as the resulting modular abelian variety has dimension greater than 1. On the other hand, formula (72) is available in that broader setting: see again [**DDLR**] for explicit examples.

The algebra $\mathbb{T}$ of Hecke operators acts by correspondences on $\mathrm{CH}^1(X_0^D(M)) \otimes \mathbb{Q}$ and gives rise via (65) to a linear action of $\mathbb{T} \otimes \mathbb{T} \otimes \mathbb{T}$ on $\mathrm{CH}^2(X_0^D(M)^3) \otimes \mathbb{Q}$ and on its subspace $\mathrm{CH}^2(X_0^D(M)^3)_0 \otimes \mathbb{Q}$ of null-homologous cycles.

Definition 3.4. — Set

$$\mathrm{MW}(g,f) := \mathrm{Hom}\left(\mathbb{T}[g] \otimes \mathbb{T}[g] \otimes \mathbb{T}[f], \mathrm{CH}^2(X_0^D(M)^3)_0 \otimes \mathbb{Q}\right)$$

and let $y_{g,f} \in \mathrm{MW}(g,f)$ be the homomorphism defined by

$$y_{g,f}(T_1 \otimes T_2 \otimes T_3) := (T_1 \otimes T_2 \otimes T_3)(\Delta_{\mathrm{GKS}}).$$

By working with a suitable integral model of $X_0^D(M)^3$, B. Gross and C. Schoen have shown in **[GrSc]** that Beilinson-Bloch's **[Bl84]** height pairing $\langle\,,\rangle^{\mathrm{BB}}$ on $\mathrm{CH}^2(X_0^D(M)^3)_0$ gives rise to a well-defined pairing

$$\langle\,,\rangle^{\mathrm{BB}} : \mathrm{MW}(g,f) \times \mathrm{MW}(g,f) \longrightarrow \mathrm{Hom}((\mathbb{T}[g] \otimes \mathbb{T}[g] \otimes \mathbb{T}[f])^{\otimes 2}, \mathbb{C}),$$

independently of any choices; see also **[YZZ**, §1.3].
X. Yuan, S. Zhang and W. Zhang have recently proved in **[YZZ]** the following formula, which was conjectured by B. Gross and S. Kudla in **[GrKu**, Conjecture 13.2]:

Theorem 3.5. — **[YZZ**, Theorem 1.3.1, Corollary 1.4.1 (1)] *Assume as in* (70) *that* $\epsilon(f) = -1$ *and* $\epsilon(\mathrm{Sym}^2(g) \otimes f) = +1$. *Then*

$$\langle y_{g,f}, y_{g,f} \rangle^{\mathrm{BB}} = L'(f,1) \cdot \prod_{\sigma : K_g \to \mathbb{C}} L(g^\sigma \otimes g^\sigma \otimes f, 2) \cdot \lambda$$

for some non-zero homomorphism $\lambda \in \mathrm{Hom}((\mathbb{T}[g] \otimes \mathbb{T}[g] \otimes \mathbb{T}[f])^{\otimes 2}, \mathbb{C})$.

Remark 3.6. — Note the different language in which the results are presented in **[YZZ]** in contrast to **[GrSc**, §8] and here: while the former exploits the language of automorphic representations, the latter adopt the more classical parlance of modular forms and Hecke operators, working with Shimura curves of fixed finite level.

The combination of Theorem 1 together with Theorem 3.5 yields the following result.

Theorem 3.7. — *Assume as in* (70) *that* $\epsilon(f) = -1$ *and* $\epsilon(\mathrm{Sym}^2(g) \otimes f) = +1$. *Then the space* $\underline{P}_{g,f} \subseteq A_f(F) \otimes \mathbb{Q}$ *has positive rank if and only if*

$$L'(f,1) \neq 0, \quad \textit{and} \quad L(\mathrm{Sym}^2(g^\sigma) \otimes f, 2) \neq 0 \ \textit{for all}\ \sigma : K_g \to \mathbb{C}. \tag{73}$$

Proof. — The automorphism $\tau : X_0^D(M)^3 \to X_0^D(M)^3$, $(x_1,x_2,x_3) \mapsto (x_2,x_1,x_3)$ gives rise to an involution τ^* on $\mathrm{CH}^2(X_0^D(M)^3)_0 \otimes \mathbb{Q}$ and on $J^2(X_0^D(M)^3)$ which preserves the (g,g,f)-isotypical components. By **[YZZ**, Corollary 1.4.1 (1)], the homomorphism $y_{g,f}$ is invariant under the action of τ.
Hence, combined with Theorem 3.5, condition (73) holds if and only if there exist correspondences $T_{g,i} \in \mathbb{T}[g]$, $T_{f,i} \in \mathbb{T}[f]$ such that,

$$\langle \Delta[\breve{g},\breve{g},\breve{f}], \Delta[\breve{g},\breve{g},\breve{f}] \rangle^{\mathrm{BB}} \neq 0,$$

where we set

$$\Delta[\breve{g},\breve{g},\breve{f}] := \sum_i T_{g,i} \otimes T_{g,i} \otimes T_{f,i}(\Delta_{\mathrm{GKS}}).$$

Lemma 2.1 combined with Lemma 3.8 below show that

$$\sum_i \Pi_{X_{12}} \cdot (T_{g,i} \otimes T_{g,i} \otimes 1)(\Delta_{\mathrm{GKS}}) = \sum_i \Pi_{T_{g,i}}(\Delta_{\mathrm{GKS}}).$$

Since $T \circ \Pi_{X_{12}} = \Pi_{X_{12}}(1 \otimes 1 \otimes T)$ for any $T \in \mathrm{CH}^1(X_0^D(M)^2)$, we obtain that

$$\sum T_{f,i} \Pi_{T_{g,i}}(\Delta_{\mathrm{GKS}}) = \Pi_{X_{12}}(\Delta[\breve{g}, \breve{g}, \breve{f}]). \tag{74}$$

For each i, $T_{f,i}$ is a self-correspondence on $X_0^D(M)$ that gives rise to an endomorphism $\pi_{f,i}$ of $\mathrm{Jac}(X_0^D(M))$ which is a linear combination of the projections π_f^d, $d \mid N/N_f$. Set

$$P[\breve{g}, \breve{g}, \breve{f}] := \sum_i \pi_{f,i} \Pi_{T_{g,i}}(\Delta_{\mathrm{GKS}})] \in A_f$$

and write $\langle\,,\rangle^{NT}$ for the Néron-Tate canonical height pairing on an abelian variety. By [**Bl84**, p. 120-121, A.2], the pairing $\langle\,,\rangle^{\mathrm{BB}}$ is functorial with respect to correspondences, and from (74) it follows that

$$\langle \Delta[\breve{g}, \breve{g}, \breve{f}], \Delta[\breve{g}, \breve{g}, \breve{f}] \rangle^{\mathrm{BB}} \doteq \langle P[\breve{g}, \breve{g}, \breve{f}], P[\breve{g}, \breve{g}, \breve{f}] \rangle^{NT},$$

up to a positive constant.
Note that, by (71), $P[\breve{g}, \breve{g}, \breve{f}]$ is a point in $\underline{P}_{g,f}$. By Theorem 1 we have $P[\breve{g}, \breve{g}, \breve{f}] = \sum_i \pi_{f,i} \kappa_{\mathrm{dR}}(\xi_{T_{g,i}})$. The corollary follows. □

Lemma 3.8. — *Given a self-correspondence $Z \in \mathrm{CH}^1(X_0^D(M)^2)$ of $X_0^D(M)$, it holds that*

$$\Pi_{X_{12}} \cdot (Z \otimes Z \otimes 1)(\Delta_{\mathrm{GKS}}) = Z \cdot X_{12} - \deg(Z \cdot X_{12}) o.$$

Proof. — By applying (46), it directly follows that $\Pi_{X_{12}} \cdot (Z \otimes Z \otimes 1)(X_{123}) = Z \cdot X_{12} \in \mathrm{Jac}(X_0^D(M))$. Similarly, when one applies (46) to the six other components X_j, X_{jk} of Δ_{GKS} one obtains that, for all $j, k = 1, 2, 3$, $j \neq k$, the divisors $\Pi_{X_{12}} \cdot (Z \otimes Z \otimes 1)(X_j)$ and $\Pi_{X_{12}} \cdot (Z \otimes Z \otimes 1)(X_{jk})$ are all linearly equivalent to some multiple of the degree 1 divisor o. Since Δ_{GKS} is null-homologous, $\Pi_{X_{12}} \cdot (Z \otimes Z \otimes 1)(\Delta_{\mathrm{GKS}})$ is a divisor class on $X_0^D(M)$ of degree 0. The lemma follows. □

In light of Theorem 1, Theorem 3.7 can be viewed to some extent as an analogue for modular and Shimura curves of the Deligne-Wojtkowiak calculations [**De**] in which values of the Riemann zeta-function $\zeta(s)$ at odd positive integers emerge from the study of the mixed Tate motives arising in the pro-unipotent fundamental group of $\mathbb{P}_1 - \{0, 1, \infty\}$. It was motivated by the authors' study of the Chow-Heegner points attached to diagonal cycles in triple products of Kuga-Sato-varieties, which is detailed more systematically in [**DR1**]. Note however that while [**De**] obtains formulae for the values of $\zeta(s)$ at certain *non-critical points*, Theorem 3.7 applies only to the *critical value* of the L-function $L(g \otimes g \otimes f, s)$ at $s = 2$.

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