

On the global and weak dimensions of
pullbacks of non-commutative rings

N. KOSMATOV

On the global and weak dimensions of pullbacks of non-commutative rings

Nikolai Kosmatov

All rings will be assumed to have identity elements preserved by ring homomorphisms, and all modules, unless specified otherwise, will be left modules. For a ring S , $\text{lgld } S$ and $\text{lwd } S$ will denote the left global dimension of S and the left weak global dimension of S , respectively. For an S -module X and a right S -module Y , the projective dimension of X , the injective dimension of X , the flat dimension of X , and the flat dimension of Y are denoted by $\text{pd}_S X$, $\text{id}_S X$, $\text{fd}_S X$ and $\text{rfd}_S Y$, respectively.

A commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ \downarrow i_2 & & \downarrow j_1 \\ R_2 & \xrightarrow{j_2} & R' \end{array} \quad (1)$$

is said to be a pullback (or a cartesian square, or a fibre product) if given $r_1 \in R_1$, $r_2 \in R_2$ with $j_1(r_1) = j_2(r_2)$ there is a unique element $r \in R$ such that $i_1(r) = r_1$ and $i_2(r) = r_2$ (note that if j_2 is a surjection then so is i_1 , but not conversely). The ring R is called the fibre product (or pullback) of R_1 and R_2 over R' .

Assuming that j_2 is surjective, Milnor [5, Chapter 2] characterized projective modules over such a ring R . Facchini and Vámos [2] established analogues of Milnor's theorems for injective and flat modules. Kirkman and Kuzmanovich [3, Theorem 2] showed that if (1) is a pullback square with j_2 surjective, then

$$\text{lgld } R \leq \max_{k=1,2} \{ \text{lgld } R_k + \text{rfd}_R R_k \}. \quad (2)$$

For commutative rings, Scrivanti [6] sharpened this upper bound on $\text{lgld } R$ and obtained an upper bound on $\text{lwd } R$. Moreover, she gave examples which

showed that, in a certain case, those results were best possible. Besides, Cowley showed that it could be beneficial to work only on one side of the rings in question [1, Example 3.4]. He obtained the following “one-sided” bound [1, Theorem 3.1]: if (1) is a pullback square with j_2 surjective, then

$$\text{lgld } R \leq \max_{k=1,2} \{\text{lgld } R_k + \text{pd}_R R_k\}. \quad (3)$$

Our Theorems 9, 10 and Corollaries 12, 13 are the generalizations to non-commutative rings of Scrivanti’s results. In Theorem 8, we provide a “one-sided” bound on the left global dimension of a pullback ring. In Propositions 5, 6 and 7, we give sufficient conditions for an R -module M to have injective, projective and flat dimensions $\leq n$. In Corollaries 11 and 12, we deduce the upper bounds (3) and (2) as immediate consequences of our Propositions 5 and 6. Here we relax the conditions and only require i_1 to be surjective.

We begin with the following consequence of [2, Theorem 2].

Theorem 1. *Let (1) be a pullback diagram with i_1 surjective. Then an R -module M is injective (projective, flat) if and only if $\text{Hom}_R(R_1, M)$ and $\text{Hom}_R(R_2, M)$ ($R_1 \otimes_R M$ and $R_2 \otimes_R M$) are R_1 - and R_2 -injective (projective, flat) modules respectively.*

Proof. Set $R'' = j_2(R_2)$. Since i_1 is a surjection, we obtain $j_1(R_1) \subset j_2(R_2) = R''$. Thus we have another commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ \downarrow i_2 & & \downarrow j_1 \\ R_2 & \xrightarrow{j_2} & R'' \end{array}$$

It is clear that this diagram is a pullback square with $j_2 : R_2 \rightarrow R''$ surjective. So the desired result follows from [2, Theorem 2].

Proposition 2. *Let M be an R -module, n be a positive integer, and let*

$$0 \longrightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \longrightarrow \dots$$

be an injective resolution of M . Let K_t denote $\text{im}(f_{t+1})$, $t \geq 0$. Suppose that $\text{id}_{R_k}(\text{Ext}_R^l(R_k, M)) \leq n - l$ for $l = 0, 1, \dots, n$ and $k = 1, 2$. Then $\text{id}_{R_k}(\text{Hom}_R(R_k, K_t)) \leq n - t - 1$ for $t = 0, 1, \dots, n$ and $k = 1, 2$.

Proof. For any $n \geq 1$ and $k = 1, 2$, the proof is by induction on t .

For $t = 0$, if we apply the functor $\text{Ext}_R^*(R_k, -)$ to the short exact sequence of R -modules $0 \rightarrow M \rightarrow I_0 \xrightarrow{f_1} K_0 \rightarrow 0$, we obtain an exact sequence of R_k -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(R_k, M) & \longrightarrow & \text{Hom}_R(R_k, I_0) & \xrightarrow{f_{1*}} & \\ & & \xrightarrow{f_{1*}} & \text{Hom}_R(R_k, K_0) & \longrightarrow & \text{Ext}_R^1(R_k, M) & \longrightarrow 0 \end{array} \quad (4)$$

and isomorphisms of R_k -modules

$$\text{Ext}_R^l(R_k, K_0) \simeq \text{Ext}_R^{l+1}(R_k, M), \quad l \geq 1. \quad (5)$$

Set $A_{k,0} = \text{im } f_{1*}$ and break up (4) into two short exact sequences of R_k -modules:

$$0 \longrightarrow \text{Hom}_R(R_k, M) \longrightarrow \text{Hom}_R(R_k, I_0) \xrightarrow{f_{1*}} A_{k,0} \longrightarrow 0, \quad (6)$$

$$0 \longrightarrow A_{k,0} \hookrightarrow \text{Hom}_R(R_k, K_0) \longrightarrow \text{Ext}_R^1(R_k, M) \longrightarrow 0. \quad (7)$$

Since I_0 is an injective R -module, it can easily be checked that R_k -module $\text{Hom}_R(R_k, I_0)$ is injective. Since $\text{id}_{R_k}(\text{Hom}_R(R_k, M)) \leq n$, we obtain from (6) that $\text{id}_{R_k}(A_{k,0}) \leq n - 1$. At the same time $\text{id}_{R_k}(\text{Ext}_R^1(R_k, M)) \leq n - 1$. Therefore, using (7), we get $\text{id}_{R_k}(\text{Hom}_R(R_k, K_0)) \leq n - 1$.

For $t \geq 1$, we apply the functor $\text{Ext}_R^*(R_k, -)$ to the short exact sequence of R -modules $0 \rightarrow K_{t-1} \hookrightarrow I_t \xrightarrow{f_{t+1}} K_t \rightarrow 0$. We get an exact sequence of R_k -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(R_k, K_{t-1}) & \longrightarrow & \text{Hom}_R(R_k, I_t) & \xrightarrow{(f_{t+1})_*} & \\ & & \xrightarrow{(f_{t+1})_*} & \text{Hom}_R(R_k, K_t) & \longrightarrow & \text{Ext}_R^1(R_k, K_{t-1}) & \longrightarrow 0 \end{array} \quad (8)$$

and isomorphisms of R_k -modules

$$\text{Ext}_R^l(R_k, K_t) \simeq \text{Ext}_R^{l+1}(R_k, K_{t-1}), \quad l \geq 1. \quad (9)$$

Put $A_{k,t} = \text{im } (f_{t+1})_*$ and break up (8) into two short exact sequences of R_k -modules:

$$0 \longrightarrow \text{Hom}_R(R_k, K_{t-1}) \longrightarrow \text{Hom}_R(R_k, I_t) \xrightarrow{(f_{t+1})_*} A_{k,t} \longrightarrow 0, \quad (10)$$

$$0 \longrightarrow A_{k,t} \hookrightarrow \text{Hom}_R(R_k, K_t) \longrightarrow \text{Ext}_R^1(R_k, K_{t-1}) \longrightarrow 0. \quad (11)$$

By the inductive hypothesis, we have $\text{id}_{R_k}(\text{Hom}_R(R_k, K_{t-1})) \leq n - t$. Since I_t is an injective R -module, $\text{Hom}_R(R_k, I_t)$ is an injective R_k -module. Therefore, from (10), $\text{id}_{R_k}(A_{k,t}) \leq n - t - 1$. Combining (5) and (9), we have $\text{Ext}_R^1(R_k, K_{t-1}) \simeq \dots \simeq \text{Ext}_R^t(R_k, K_0) \simeq \text{Ext}_R^{t+1}(R_k, M)$. Hence $\text{id}_{R_k}(\text{Ext}_R^1(R_k, K_{t-1})) = \text{id}_{R_k}(\text{Ext}_R^{t+1}(R_k, M)) \leq n - t - 1$. Finally, from (11), we obtain $\text{id}_{R_k}(\text{Hom}_R(R_k, K_t)) \leq n - t - 1$, as required.

The proofs of the following Propositions 3 and 4 are similar to that of Proposition 2 if we apply the functors $\text{Tor}_*^R(R_k, -)$ to the given resolutions.

Proposition 3. *Let M be an R -module, n be a positive integer, and let*

$$\dots \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a projective resolution of M . Let K_t denote $\ker f_t$, $t \geq 0$. Suppose that $\text{pd}_{R_k}(\text{Tor}_l^R(R_k, M)) \leq n - l$ for $l = 0, 1, \dots, n$ and $k = 1, 2$. Then $\text{pd}_{R_k}(R_k \otimes_R K_t) \leq n - t - 1$ for $t = 0, 1, \dots, n$ and $k = 1, 2$.

Proposition 4. *Let M be an R -module, n be a positive integer, and let*

$$\dots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a flat resolution of M . Let K_t denote $\ker f_t$, $t \geq 0$. Suppose that $\text{fd}_{R_k}(\text{Tor}_l^R(R_k, M)) \leq n - l$ for $l = 0, 1, \dots, n$ and $k = 1, 2$. Then $\text{fd}_{R_k}(R_k \otimes_R K_t) \leq n - t - 1$ for $t = 0, 1, \dots, n$ and $k = 1, 2$.

Proposition 5. *Let (1) be a pullback diagram with i_1 surjective, M be an R -module, and let n be a non-negative integer. Suppose that $\text{id}_{R_k}(\text{Ext}_R^l(R_k, M)) \leq n - l$ for $l = 0, 1, \dots, n$ and $k = 1, 2$. Then $\text{id}_R M \leq n$.*

Proof. For the case $n = 0$, the result follows from Theorem 1. For $n \geq 1$, consider an injective resolution of R -module M

$$0 \longrightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \longrightarrow \dots$$

By definition, put $K_t = \text{im}(f_{t+1})$ for $t \geq 0$. From Proposition 1, we get that R_k -module $\text{Hom}_R(R_k, K_{n-1})$ is injective ($k = 1, 2$). Hence, by Theorem 1, K_{n-1} is an injective R -module. This means that $\text{id}_R M \leq n$.

Arguing as above, the reader will easily prove the following analogues of Proposition 5 for projective and flat dimensions.

Proposition 6. *Let (1) be a pullback diagram with i_1 surjective, M be an R -module, and let n be a non-negative integer. Suppose that $\text{pd}_{R_k}(\text{Tor}_l^R(R_k, M)) \leq n - l$ for $l = 0, 1, \dots, n$ and $k = 1, 2$. Then $\text{pd}_R M \leq n$.*

Proposition 7. *Let (1) be a pullback diagram with i_1 surjective, M be an R -module, and let n be a non-negative integer. Suppose that $\text{fd}_{R_k}(\text{Tor}_l^R(R_k, M)) \leq n - l$ for $l = 0, 1, \dots, n$ and $k = 1, 2$. Then $\text{fd}_R M \leq n$.*

Proposition 5 clearly implies the following theorem.

Theorem 8. *Let (1) be a pullback diagram with i_1 surjective, and let n be a non-negative integer. Suppose that for any R -module M we have that*

$$\text{id}_{R_k}(\text{Ext}_R^l(R_k, M)) \leq n - l \text{ for } l = 0, 1, \dots, n \text{ and } k = 1, 2.$$

Then $\text{lgld } R \leq n$.

Theorem 9. *Let (1) be a pullback diagram with i_1 surjective, and let n be a non-negative integer. Suppose that for any left ideal J of R we have that*

$$\text{pd}_{R_k}(\text{Tor}_l^R(R_k, R/J)) \leq n - l \text{ for } l = 0, 1, \dots, n \text{ and } k = 1, 2.$$

Then $\text{lgld } R \leq n$.

Proof. Let J be a left ideal of R . Proposition 6 shows immediately that $\text{pd}_R(R/J) \leq n$. Therefore, using Auslander's theorem, we have $\text{lgld } R = \sup\{\text{pd}_R(R/J) \mid J \text{ is a left ideal of } R\} \leq n$.

Theorem 10. *Let (1) be a pullback diagram with i_1 surjective, and let n be a non-negative integer. Suppose that for any finitely generated left ideal J of R we have that*

$$\text{fd}_{R_k}(\text{Tor}_l^R(R_k, R/J)) \leq n - l \text{ for } l = 0, 1, \dots, n \text{ and } k = 1, 2.$$

Then $\text{lwd } R \leq n$.

Proof. Let J be a finitely generated left ideal of R . Proposition 7 evidently implies that $\text{fd}_R(R/J) \leq n$. Therefore, since $\text{lwd } R = \sup\{\text{pd}_R(R/J) \mid J \text{ is a finitely generated left ideal of } R\}$, we have $\text{lgld } R \leq n$.

Corollary 11. *If (1) is a pullback diagram with i_1 surjective, then*

$$\text{lgld } R \leq \max_{k=1,2} \{\text{lgld } R_k + \text{pd}_R R_k\}.$$

Proof. Set $n_k = \text{lgld } R_k$, $m_k = \text{pd}_R R_k$, $N_k = n_k + m_k$ ($k = 1, 2$) and $N = \max\{N_1, N_2\}$. It can be assumed that $m_k, n_k < \infty$. Let M be an R -module and $k \in \{1, 2\}$. Since $\text{pd}_R R_k = m_k$, we have $\text{Ext}_R^l(R_k, M) = 0$ for all $l \geq m_k + 1$. At the same time, since $\text{lgld } R_k = n_k$, we get $\text{id}_{R_k}(\text{Ext}_R^l(R_k, M)) \leq n_k = N_k - m_k \leq N_k - l \leq N - l$ for any $l = 0, 1, \dots, m_k$. Therefore, by Proposition 5, $\text{pd}_R M \leq N$. This means that $\text{lgld } R \leq N$.

Similarly, Propositions 6 and 7 allow us to prove Corollaries 12 and 13.

Corollary 12. *If (1) is a pullback diagram with i_1 surjective, then*

$$\text{lgld } R \leq \max_{k=1,2} \{\text{lgld } R_k + \text{rfd}_R R_k\}.$$

Corollary 13. *If (1) is a pullback diagram with i_1 surjective, then*

$$\text{lwd } R \leq \max_{k=1,2} \{\text{lwd } R_k + \text{rfd}_R R_k\}.$$

Acknowledgement. The authour would like to thank Professor A. I. Generalov for his help and guidance.

References

- [1] K. M. COWLEY. *One-sided bounds and the vanishing of Ext*. J. Algebra, **190** (1997), 361–371.
- [2] A. FACCHINI, P. VÁMOS. *Injective modules over pullbacks*. J. London Math. Soc., **31** (1985), 425–438.
- [3] E. KIRKMAN, J. KUZMANOVICH. *On the global dimension of fibre products*. Pacific J. Math., **134** (1988), 121–132.
- [4] S. MACLANE. *Homology*. Springer-Verlag, New York, 1963.

- [5] J. MILNOR. *Introduction to Algebraic K-theory*. Annals Math. Studies, vol. **72**, Princeton University Press, Princeton, 1971.
- [6] S. SCRIVANTI. *Homological dimension of pullbacks*. Math. Scand., **71** (1992), 5–15.

Equipe de Mathématiques
U.M.R. 6623 du C.N.R.S.
16, route de Gray
25030 Besançon Cedex France

Department of Mathematics and Mechanics
Saint-Petersburg State University
Bibliotechnaya pl. 2
Saint-Petersburg, 198904, Russia
E-Mail: koko@nk1442.spb.edu