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Carmelo Antonio Finocchiaro and Marco Fontana

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Centre international de rencontres mathématiques
U.M.S. 822 C.N.R.S./S.M.F.
Luminy (Marseille) France

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Some applications of the ultrafilter topology on spaces of valuation domains, Part II

Carmelo Antonio Finocchiaro and Marco Fontana

Abstract

Let K be a field and A be a subring of K. In the present note, we present the main applications of the so called *ultrafilter topology* on the space $\mathrm{Zar}(K|A)$, introduced in the previous Part I. After recalling that $\mathrm{Zar}(K|A)$ is a spectral space, we give an explicit description of $\mathrm{Zar}(K|A)$ as the prime spectrum of a ring (even in the case when the quotient field of A is a proper subfield of K). Moreover, we provide applications of the topological material previously introduced to the study of representations of integrally closed domains and valuative semistar operations.

Introduction

Let K be a field and A be a subring of K. Denote by $\mathrm{Zar}(K|A)$ the set of all the valuation domains having K as the quotient field, and containining A as a subring; denote simply by $\mathrm{Zar}(K)$ the set of all the valuation domains of K. As usual, $\mathrm{Zar}(K|A)$ can be equipped with the so called *Zariski topology*, that is, the topology having a basis for the open sets given by the family of all the subsets of the type $B_F := \{V \in \mathrm{Zar}(K|A) \mid V \supseteq F\}$, where F is a finite subset of K. It is well-known that $\mathrm{Zar}(K|A)$ is quasi-compact, and it is Hausdorff only in the trivial case. A natural way to make $\mathrm{Zar}(K|A)$ a compact Hausdorff topological space is to consider on it the so called *ultrafilter topology* (see [2] and [3]). In the present Note, we will provide applications of the topological properties of $\mathrm{Zar}(K|A)$, endowed with the ultrafilter topology (or, with the inverse topology, in the sense of Hochster [6]), to the representations of integrally closed domains as intersections of valuation overrings. Moreover, we will also apply some of our results to characterize when two valuative semistar operations have their associated finite type semistar operations equal.

1. Identifying $\mathrm{Zar}(K|A)$ with a "nice" spectral space

Let K be a field and T be an indeterminate over K. Recall that a subring S of $K(T)$ is called a $K-$*function ring* if T is invertible in S and $\frac{f(0)}{f(T)} \in S$, for each nonzero polynomial $f(T) \in K[T]$. This notion was introduced by Halter–Koch in [5] as a generalization of the classical construction of the Kronecker function ring. We collect in the following remark the basic algebraic properties of $K-$function rings.

Remark 1.1. (see [5]) Let K be a field and T be an indeterminate over K.

(i) The intersection of a nonempty collection of $K-$function rings is a $K-$function ring.
(ii) Each $K-$function ring is a Bézout domain with quotient field $K(T)$.
(iii) If V is a valuation domain of K and $\mathfrak{m}$ is the maximal ideal of V, then the localization $V(T) := V[T]_{\mathfrak{m}[T]}$ (usually called *the trivial extension of V in $K(T)$*) is both a valuation domain of $K(T)$ and a $K-$function ring.

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Given a subring S of $K(T)$, we will denote by $\mathrm{Zar}_0(K(T)|S)$ the subset of $\mathrm{Zar}(K(T)|S)$ consisting of all the valuation domains of $K(T)$ that are trivial extensions of some valuation domain of K.

The following result provides a characterization of $K-$function rings.

Proposition 1.2. *Let K be a field, T an indeterminate over K and S a subring of $K(T)$. Then, the following conditions are equivalent.*

(i) *S is a $K-$function ring.*
(ii) *S is integrally closed and $\mathrm{Zar}(K(T)|S) = \mathrm{Zar}_0(K(T)|S)$.*
(iii) *S is the intersection of a nonempty subcollection of $\mathrm{Zar}_0(K(T))$.*

Proposition 1.3. *Let K be a field and T an indeterminate over K. The following statements hold.*

(a) *The natural map $\varphi : \mathrm{Zar}(K(T)) \longrightarrow \mathrm{Zar}(K)$, $W \mapsto W \cap K$, is continuous and closed with respect to both Zariski topologies or ultrafilter topologies.*
(b) *If $S \subseteq K(T)$ is a $K-$function ring, then the restriction of φ to the subspace $\mathrm{Zar}(K(T)|S)$ of $\mathrm{Zar}(K(T))$ is a topological embedding, with respect to both Zariski topologies or ultrafilter topologies.*
(c) *Let A be* any *subring of K, and let*

$$\mathrm{Kr}(K|A) := \bigcap\{V(T) \mid V \in \mathrm{Zar}(K|A)\}.$$

Then $\mathrm{Kr}(K|A)$ is a $K-$function ring. Moreover, the restriction of φ to $\mathrm{Zar}(K(T)|\mathrm{Kr}(K|A))$ establishes a homeomorphism of $\mathrm{Zar}(K(T)|\mathrm{Kr}(K|A))$ with $\mathrm{Zar}(K|A)$, with respect to both Zariski topologies or ultrafilter topologies.
(d) *Let A be a subring of K, $S := \mathrm{Kr}(K|A)$, and let $\gamma : \mathrm{Zar}(K(T)|S) \longrightarrow \mathrm{Spec}(S)$ be the map sending a valuation overring of S into its center on S. Then γ establishes a homeomorphism, with respect to both Zariski topologies or ultrafilter topologies; thus, the map*

$$\gamma \circ \varphi^{-1} : \mathrm{Zar}(K|A) \longrightarrow \mathrm{Zar}(K(T)|S) \longrightarrow \mathrm{Spec}(S)$$

is also a homeomorphism. In other words, $\mathrm{Zar}(K|A)$ is a spectral space, endowed with both Zariski topology or ultrafilter topology.

2. Some applications

The first application that we give is a topological interpretation of when two given collections of valuation domains are representations of the same integral domain.

Proposition 2.1. *Let K be a field. If Y_1, Y_2 are nonempty subsets of $\mathrm{Zar}(K)$ having the same closure in $\mathrm{Zar}(K)$, with respect to the ultrafilter topology, then*

$$\bigcap\{V \mid V \in Y_1\} = \bigcap\{V \mid V \in Y_2\}.$$

In particular, if $\mathtt{Cl}^{ultra}(Y)$ denotes the closure of a nonempty subset Y of $\mathrm{Zar}(K)$ with respect to the ultrafilter topology, then

$$\bigcap\{V \mid V \in Y\} = \bigcap\{V \mid V \in \mathtt{Cl}^{ultra}(Y)\}.$$

The converse of the previous statement is false. In fact, we will see that equality of the closures of the subsets Y_1, Y_2, with respect to the ultrafilter topology, implies a statement that, in general, is stronger than the equality of the (integrally closed) domains obtained by intersections. To see this, recall some background material about semistar operations.

Let A be an integral domain, K be the quotient field of A. As usual, denote by $\overline{\boldsymbol{F}}(A)$ the set of all nonzero $A-$submodule of K, and by $\boldsymbol{f}(A)$ the set of all nonzero finitely generated $A-$submodule of K. As it is well known, a nonempty subset Y of $\mathrm{Zar}(K|A)$ induces the *valuative semistar operation* $\wedge_Y$, defined by $F^{\wedge_Y} := \bigcap\{FV \mid V \in Y\}$, for each $F \in \overline{\boldsymbol{F}}(A)$. A valuative semistar operation $\star$ is always `e.a.b.`, that is, for all $F, G, H \in \boldsymbol{f}(A)$, $(FG)^\star \subseteq (FH)^\star$ implies $G^\star \subseteq H^\star$. Recall that we can associate to any semistar operation $\star$ on A a semistar operation $\star_f$ *of finite type* (on A), by setting $F^{\star_f} := \bigcup\{G^\star \mid G \in \boldsymbol{f}(A),\ G \subseteq F\}$, for each $F \in \overline{\boldsymbol{F}}(A)$; $\star_f$ is called *the semistar operation of finite type associated to $\star$.*

For any subset $Y \subseteq \mathrm{Zar}(K|A)$, denote by $Y^{\uparrow}$ *the Zariski–generic closure of* Y, that is, $Y^{\uparrow} := \{W \in \mathrm{Zar}(K|A) \mid W \subseteq V, \text{ for some } V \in Y\}$.

Theorem 2.2. *Let A be an integral domain, K its quotient field, and Y_1, Y_2 two nonempty subsets of* $\mathrm{Zar}(K|A)$*. Then, the following conditions are equivalent.*

(i) *The semistar operations of finite type associated to $\wedge_{Y_1}$ and $\wedge_{Y_2}$ are the same, that is, $(\wedge_{Y_1})_f = (\wedge_{Y_2})_f$.*

(ii) *The subsets $\mathtt{Cl}^{ultra}(Y_1)$, $\mathtt{Cl}^{ultra}(Y_2)$ of* $\mathrm{Zar}(K|A)$ *have the same Zariski–generic closure, that is, $\mathtt{Cl}^{ultra}(Y_1)^{\uparrow} = \mathtt{Cl}^{ultra}(Y_2)^{\uparrow}$.*

Let A be an integral domain, K its quotient field and $Z := \mathrm{Zar}(K|A)$. For any nonempty subset $Y \subseteq Z$, consider the $K-$function ring $\mathrm{Kr}(Y) := \bigcap\{V(T) \mid V \in Y\}$. We say that A is *a vacant domain* if it is integrally closed and, for any representation Y of A, we have $\mathrm{Kr}(Y) = \mathrm{Kr}(Z)$ (see [1]).

Corollary 2.3. *Let A be an integrally closed domain and K its quotient field. The following conditions are equivalent.*

(i) *A is a vacant domain.*

(ii) *For any representation Y of A, $\mathtt{Cl}^{ultra}(Y)^{\uparrow} = \mathrm{Zar}(K|A)$.*

Recall that a semistar operation is *complete* if it is `e.a.b.` and of finite type (see [4] for further equivalent definitions of complete semistar operation). The following result provides a topological characterization of when a semistar operation is complete.

Theorem 2.4. *Let A be an integral domain, K its quotient field and $\star$ a semistar operation on A. Then, the following conditions are equivalent.*

(i) *$\star$ is complete.*

(ii) *There is a quasi-compact subspace Y of* $\mathrm{Zar}(K|A)$*, equipped with the ultrafilter topology, such that $\wedge_Y = \star$.*

(iii) *There is a quasi-compact subspace Y' of* $\mathrm{Zar}(K|A)$*, equipped with the Zariski topology, such that $\wedge_{Y'} = \star$.*

(iv) *There is a quasi-compact and Zariski–generically closed subspace Y'' of* $\mathrm{Zar}(K|A)$*, equipped with the ultrafilter topology, such that $\wedge_{Y''} = \star$.*

Corollary 2.5. *Let A be an integral domain, K its quotient field, and Y a nonempty subset of* $\mathrm{Zar}(K|A)$*. Then $(\wedge_Y)_f = \wedge_{\mathtt{Cl}^{ultra}(Y)}$.*

Following [6], the spectral space $\mathrm{Zar}(K|A)$ can be also considered with the so called *inverse (*or *dual) topology*, that is the topology for which a basis for the closed sets consists of all the subspaces of $\mathrm{Zar}(K|A)$ that are quasi-compact and open with respect to the Zariski topology. Keeping in mind that it is known that the ultrafilter topology and the constructible topology on $\mathrm{Zar}(K|A)$ coincide (see [3]), the following result follows easily.

Proposition 2.6. *Let K be a field and A be a subring of K. For any subset Y of* $\mathrm{Zar}(K|A)$*, denote by $\mathtt{Cl}^{inv}(Y)$ the closure of Y with respect to the inverse topology. Then $\mathtt{Cl}^{inv}(Y) = \mathtt{Cl}^{ultra}(Y)^{\uparrow}$.*

Finally, we can formulate some previous results in terms of Hochster's inverse topology.

Corollary 2.7. *Let A be an integral domain, K be its quotient field. The following statements hold.*

(a) *If Y_1, Y_2 are nonempty subsets of* $\mathrm{Zar}(K|A)$*, then $(\wedge_{Y_1})_f = (\wedge_{Y_2})_f$ if and only if Y_1, Y_2 have the same closure with respect to the inverse topology.*

(b) *A is a vacant domain if and only if it is integrally closed and any representation Y of A is dense in* $\mathrm{Zar}(K|A)$ *with respect to the inverse topology.*

(c) *For any nonempty subset Y of* $\mathrm{Zar}(K|A)$*, $(\wedge_Y)_f = \wedge_{\mathtt{Cl}^{inv}(Y)}$.*

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C.A.F. - Dipartimento di Matematica
Università degli studi Roma Tre
Largo San Leonardo Murialdo 1, 00146 Roma, Italy • carmelo@mat.uniroma3.it

M.F. - Dipartimento di Matematica
Università degli studi Roma Tre
Largo San Leonardo Murialdo 1, 00146 Roma, Italy • fontana@mat.uniroma3.it