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The category of cofinite modules for ideals of dimension one and codimension one

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The category of cofinite modules for ideals of dimension one and codimension one

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We assume that all rings are commutative and noetherian with identity throughout this paper. In this paper, we shall introduce several results on the category $\mathcal{M}(A, I)_{cof}$ (See Definition 1 below) for ideals I of dimension one and codimension one (cf. [11] and [9]).

1. Introduction

In this section, we introduce former results on our research and several definitions. The following theorem is fundamental, due to Matlis and Grothendieck (cf. [13] and [3]).

Theorem A. Let A be a complete local ring, with maximal ideal $\mathfrak{m}$, and residue field $k = A/\mathfrak{m}$. Let $E = E_A(k)$ be an injective hull of k over A. For an A-module N, the following conditions are equivalent.

(i) N satisfies the descending chain conditions (dcc);
(ii) N is a submodule of E^n, the direct sum of n copies of E, for some n;
(iii) There is an A-module M of finite type such that N is isomorphic to $\mathrm{Hom}_A(M, E)$;
(iv) $\mathrm{Supp}_A N \subseteq V(\mathfrak{m})$ and $\mathrm{Hom}_A(k, N)$ is of finite type;
(v) $\mathrm{Supp}_A N \subseteq V(\mathfrak{m})$ and $\mathrm{Ext}^i_A(k, N)$ is of finite type for all i;
(vi) $\mathrm{Supp}_A N \subseteq V(\mathfrak{m})$ and $\mathrm{Hom}_A(N, E)$ is of finite type.

Proof. See [5] for the proof (See [8] also). □

Next recall several definitions. Let $\mathcal{M}(A)$ be the category of all modules over a ring A.

Definition 1 (I-cofiniteness on modules). Let $\mathcal{M}(A, I)_{cof}$ be the class of modules N over a ring A satisfying the condition

$(*)$ $\mathrm{Supp}_A(N) \subseteq V(I)$ and $\mathrm{Ext}^j_A(A/I, N)$ is of finite type, for all j,

where I is an ideal of A. The objects of $\mathcal{M}(A, I)_{cof}$ are called I-cofinite.

Definition 2 (Abelian category). Let A, I and $\mathcal{M} = \mathcal{M}(A, I)_{cof}$ be as above. The full subcategory $\mathcal{M}$ is called Abelian, if it is closed under the kernel and cokernel of a morphism (See [6, p. 202] for the definition of Abelian category).

Definition 3 (Derived categories and Thick subcategories (cf. [7] and [12])). Let $\mathcal{D}^*(A)$ be the derived category, whose objects are complexes consisting of A-modules, where we write * in place of $+$, $-$, b or $\emptyset$. Further let A' be a thick Abelian subcategory of $\mathcal{M}(A)$, that is any extension in $\mathcal{M}(A)$ of two objects of A' is in A'. We define $\mathcal{D}^*_{A'}(A)$ to be the full subcategory of $\mathcal{D}^*(A)$

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consisting of those complexes $X^\bullet$ whose cohomology objects $H^i(X^\bullet)$ are all in A'. In this paper, we denote $\mathcal{D}^*_{ft}(A)$ for $\mathcal{D}^*_{A'}(A)$ in the case that A' is the category consisting of all A-modules of finite type, following the notations of [5].

Definition 4 (I-dualizing functor). Let A be a ring equipped with a dualizing complex $\mathbf{D}$, I an ideal of A. Let $\Gamma_I(-)$ be the I-power torsion subfunctor of the identity functor on $\mathcal{M}(A)$ (cf. [12, §1]). Set $D_I(-)$ to be the functor $\mathbb{R}\operatorname{Hom}^\bullet(-, \mathbb{R}\Gamma_I(\mathbf{D}))$ on the derived category $\mathcal{D}(A)$. In this paper, we call this functor $D_I(-)$ the ***I*-dualizing functor** (See [12, § 4.3]).

Definition 5 (I-cofiniteness on complexes). Let A and I be as above. Let $N^\bullet$ be an object of the derived category $\mathcal{D}(A)$. We say that $N^\bullet$ is ***I*-cofinite**, if there exists $M^\bullet \in \mathcal{D}_{ft}(A)$, such that $N^\bullet \simeq D_I(M^\bullet)$ in $\mathcal{D}(A)$. Here $D_I(-)$ is the I-dualizing functor.

Here we recall the affine duality theorem and a characterization of cofinite complexes (See [5] for the proofs):

Theorem B (Affine duality theorem). Let R be a regular ring of finite Krull dimension d and J an ideal of R. Suppose that R is complete with respect to J-adic topology. Then the natural morphism of functors $id \to D_J \circ D_J$ is an isomorphism, for complexes in either of the categories $\mathcal{D}_{ft}(R)$ or $\mathcal{D}(R,J)_{cof}$, where we denote by $\mathcal{D}(R,J)_{cof}$ the essential image of $\mathcal{D}_{ft}(R)$ by $D_J(-)$.

Theorem C (Characterization of cofinite complexes). Let R and J be as above, $N^\bullet$ in $\mathcal{D}^+(R)$. Suppose that R is complete with respect to the J-adic topology. Then $N^\bullet$ is J-cofinite if and only if

(a) $\operatorname{Supp} H^i(N^\bullet) \subseteq V(J)$ for each i, and
(b) $\operatorname{Ext}^j(R/J, N^\bullet)$ is of finite type over R, for each j.

It is natural to ask whether Theorem A holds for non-maximal ideals of A. Four questions were proposed in the paper [5, §2]. In particular the following are given:

Question 1 (Second Question). Let J be an ideal of a regular ring R of finite Krull dimension. Does the class $\mathcal{M}(R,J)_{cof}$ form an Abelian full subcategory of $\mathcal{M}(R)$?

Question 2 (Fourth Question). Does there exist an Abelian category $\mathcal{M}_{cof}$ consisting of R-modules, such that objects $N^\bullet \in \mathcal{D}(R,J)_{cof}$ are characterized by the property "$H^i(N^\bullet) \in \mathcal{M}_{cof}$" for all i?

In [5, §3 An Example], Question 1 and Question 2 are answered negatively for an ideal of dimension two. The example is as follows: Let R be the formal power series ring $k[x,y][[u,v]]$ over a polynomial ring $k[x,y]$, where k is a field and J the ideal (u,v) of R. Let M be the R-module $R/(xv+yu)$. Then it is proved that the local cohomology module $H^2_J(M)$ is not J-cofinite in [5, §3 An Example]. Even the socle $\operatorname{Hom}_R(k, H^2_J(M))$ is not finitely generated. The ideal J is of dimension two and not principal, and there is an exact sequence:

$$0 \longrightarrow H^1_J(M) \longrightarrow H^2_J(R) \longrightarrow H^2_J(R) \longrightarrow H^2_J(M) \longrightarrow 0.$$

Since J is generated by a regular sequence u, v, the local cohomology module $H^2_J(R)$ is J-cofinite. If Question 1 is affirmatively answered for the ideal J, then the local cohomology module $H^2_J(M)$ must be J-cofinite, which is false for this example. Further, if Question 2 is affirmatively answered for the ideal J, then $\operatorname{Hom}_R(R/J, H^2_J(M))$ must be of finite type by the local duality theorem (cf. [5, Theorem 2.1]) and the characterization of cofinite complexes, which gives a contradiction.

2. The cases for ideals of dimension one over local rings

Now we shall introduce the following theorems:

Theorem 1 (cf. [11, Theorem 1]). Let $(A, \mathfrak{m})$ be a local ring, and I an ideal of A. If I is an ideal of A of dimension one, then $\mathcal{M}(A,I)_{cof}$ is an Abelian full subcategory of $\mathcal{M}(A)$.

Theorem 2 (cf. [11, Theorem 2]). Let $(R, \mathfrak{n})$ be a regular local ring, and J an ideal of R of dimension one. Let $N^\bullet$ be in the derived category $\mathcal{D}^+(R)$ and suppose that R is complete with respect to the J-adic topology. Then $N^\bullet$ is J-cofinite if and only if $H^i(N^\bullet)$ is in $\mathcal{M}(R,J)_{cof}$ for all i.

Remark 1. Recently Theorem 2 is extended to complete Gorenstein domains, using the refined Lemmas from those of Huneke-Koh [8] (cf. [1, Theorem 1]).

Delfino and Marley proved that $\mathcal{M}(A,P)_{cof}$ is an Abelian full subcategory of $\mathcal{M}(A)$ for a prime ideal P of dimension one over a complete local ring A (cf. [2, Theorem 2]). Melkersson proved some related results (cf. [14, Theorem 7.4, Theorem 7.6, Theorem 7.7]).

3. The cases for ideals of codimension one over rings

The following result from [9] may have been known before, though the author has been unable to find it in the literature.

Theorem 3 (cf. [9]). Let A be a noetherian ring, and I an ideal of A. If I is an ideal generated by one element x of A up to radical, then $\mathcal{M}(A,I)_{cof}$ is an Abelian full subcategory of $\mathcal{M}(A)$.

Remark 2. Let M be a non zero module in $\mathcal{M}(A,I)_{cof}$. If $\sqrt{I}=\sqrt{(x)}$ and x is not a unit, then x^n is a zero divisor on M for some n, since Supp$M$ is contained in $V(x)$. Further it holds that $\Gamma_I(M)=M$.

The following also holds from Theorem 3, since the height one prime ideal is principal in a unique factorization domain.

Corollary 1. Let R be a unique factorization domain, and J an ideal of pure height one. Then $\mathcal{M}(R,J)_{cof}$ is an Abelian full subcategory of $\mathcal{M}(R)$.

Finally, the author conjectures that Theorem 1 may be true without the hypothesis that the ring be local, though this has not yet been proved:

Conjecture. Let A be a noetherian ring, which is not local, and I an ideal of A. If I is an ideal of dimension one, then the category $\mathcal{M}(A,I)_{cof}$ is Abelian.

On the other hand, the author suspects that $\mathcal{M}(A,I)_{cof}$ is a Serre subcategory of $\mathcal{M}(A)$, for an ideal I of dimension one. But he has no counterexample.

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